Level-0 action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the q-deformed Fock spaces

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Abstract

On the level-1 Fock space modules of the algebra $U_q(\widehat{\mathfrak{sl}}_n)$ we define a level-0 action U_0 of the $U_q(\widehat{\mathfrak{sl}}_n)$, and an action of an abelian algebra of conserved Hamiltonians commuting with the U_0. An irreducible decomposition of the Fock space with respect to the level-0 action is derived by constructing a base of the Fock space in terms of the Non-symmetric Macdonald Polynomials.

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1 Introduction

Recently several intriguing links have been uncovered between Solvable Models with Long-Range Interaction and representations of affine Lie algebras.

The earliest example is the identification of the Field Theory limit in the Haldane-Shastry spin chain with the level-1 $su(2)$ WZW Conformal Field Theory made in [11]. This led to discoveries such as the Yangian action on the integrable level-1 modules of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$, the spinon bases and the fermionic character formulas [4, 5].

Another line of research connected the Calogero-Sutherland Model and its q-deformed analog – the trigonometric Ruijsenaars Model – with the W-algebras and their q-deformations [9].

In the work of Bernard *et al.* [3] both the Haldane-Shastry and the Calogero-Sutherland Models were understood to be special cases of the more general $\mathfrak{sl}_n$-invariant Dynamical or Spin Calogero-Sutherland Model. In this light it becomes plausible that a connection between the Long-Range Models and CFT must exist on the level of the Spin Calogero-Sutherland Model as well, so that the two lines of results mentioned above can be seen as parts of a more general structure.

Two attempts to understand the Spin Calogero-Sutherland Model in the Field Theory limit were made by different means in [10] and in [16]. In [16] this limit was constructed for the $\mathfrak{sl}_2$-invariant case by using the semi-infinite wedge realization of the Fock space module of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ [12]. The Fock space was interpreted as the space of states of the Model in the Field Theory limit, it was shown to admit a Yangian action and the decomposition of this space with respect to this action was derived.

The aim of the present paper is to give a q-deformation of the construction in [16] in the $\mathfrak{sl}_n$-invariant situation. Let us briefly go through the main features of our work. We start with the q-deformation of the N-particle fermionic Spin Calogero-Sutherland Model which was proposed in [3] (see also [14]). The ingredients that define the Model are: a family of mutually commuting operators – conserved Hamiltonians $h_l^{(N)}$, $(l = \pm 1, \pm 2, \dots)$ and the level-0 action of the q-deformed affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ which we denote by $U_0^{(N)}$. This action commutes with the operators $h_l^{(N)}$ and hence has a meaning of a non-abelian symmetry algebra of the Model. The space of states in the Model is identified with the finite N-fold q-wedge product which was recently introduced in [1], it is isomorphic to the space of q-fermionic states proposed in [3].

Our main problem can be formulated as that of taking the number of particles N in the Model to infinity so that the commuting Hamiltonians and the symmetry algebra remain well-defined. To solve this problem we utilize a certain projective limit of the finite q-wedge product – the semi-infinite q-wedge product of [1].

The semi-infinite q-wedge product is shown in [1] to be isomorphic to the level-1 q-deformed Fock space module of $U_q(\widehat{\mathfrak{sl}}_n)$ discovered in [2]. We are able to demonstrate that the Hamiltonians and the commuting with them level-0 action of $U_q(\widehat{\mathfrak{sl}}_n)$ carry over from the finite into the semi-infinite q-wedge product and therefore define the Model in the Fock space. By analogy with the rational case considered in [16] we interpret this as a Field Theory limit of the q-deformed Spin Calogero-Sutherland Model and of the associated symmetry algebra.

The second problem which we consider is to derive the decomposition of the Fock space with respect to the level-0 action, and diagonalize the commuting Hamiltonians. Towards this end we construct a suitable base of the Fock space in terms of the Non-symmetric Macdonald Polynomials [15, 7]. One of the features of the level-0 decomposition of the Fock space is its relative simplicity – at generic values of parameters in the Model each irreducible component is isomorphic to a tensor product of *fundamental* representations of $U_q(\widehat{\mathfrak{sl}}_n)$ in terminology of [6] – this makes it different from the level-0 decomposition of the irreducible level-1 $U_q(\widehat{\mathfrak{sl}}_2)$ modules constructed in [13] where an irreducible component is in general a subquotient of a tensor product of the fundamental representations.

The arrangement of the paper is as follows. The sec.2 is has an introductory character, here we give the relevant background information on the affine Hecke algebra and q-wedge products. In sec. 3 the space of finite wedges is decomposed with respect to the $U_q(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(N)}$. In sec. 4 we define the level-0 action and the family of commuting Hamiltonians in the Fock space. Sec. 5 contains results on the decomposition of the Fock space with respect to this action.

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2 Preliminaries

In this introductory section we set up our notations and collect several known definitions and results to be used starting from sec. 3. For details one may consult the works [1, 3, 7, 15].

2.1 Affine Hecke algebra

The affine Hecke algebra $\widehat{H}_N(q)$ is an associative algebra generated by elements T_i ($i = 1, \ldots, N-1$) and $y_j^{\pm 1}$ ($j = 1, \ldots, N$). These elements satisfy the following relations:

$$\begin{aligned}
T_i^2 &= (q - q^{-1})T_i + I, && (2.1)\\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && (2.2)\\
T_i T_j &= T_j T_i \qquad \text{if } |i-j| > 1, && (2.3)\\
y_i y_j &= y_j y_i, && (2.4)\\
y_i T_j &= T_j y_i \qquad \text{if } i \neq j, j+1, && (2.5)\\
T_i y_i &= y_{i+1} T_i^{-1}. && (2.6)
\end{aligned}$$

We consider two different actions of $\widehat{H}_N(q)$ in $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$.
The first of these actions is defined by:

$$y_i = z_i^{-1}, \quad \text{and} \quad T_i = g_{i,i+1}, \tag{2.7}$$

where the operators $g_{i,j}$ are as follows [3] :

$$g_{i,j} = \frac{q^{-1} z_i - q z_j}{z_i - z_j}(K_{i,j} - I) + q, \quad 1 \leq i \neq j \leq N, \tag{2.8}$$

and $K_{i,j}$ is the permutation operator for variables z_i and z_j.
The other action of $\widehat{H}_N(q)$ is specified by:

$$y_i = q^{1-N} Y_i^{(N)}, \quad \text{and} \quad T_i = g_{i,i+1}, \tag{2.9}$$

here $Y_i^{(N)}$ are difference operators:

$$Y_i^{(N)} = K_{i,i+1} g_{i,i+1}^{-1} \ldots K_{i,N} g_{i,N}^{-1} p^{D_i} K_{1,i} g_{1,i} \ldots K_{i-1,i} g_{i-1,i}, \tag{2.10}$$

and

$$p^{D_i} f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, p z_i, \ldots, z_N), \quad f \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}].$$

Throughout this paper the q is taken to be a complex number which is not a root of unity ($q = 1$ is allowed). We call such q and a $p \in \mathbb{C} \setminus q^{2\mathbb{Q}_{\geq 0}}$ generic, and in what follows we consider only generic q and p unless stated otherwise.

2.2 Eigenfunctions of the operators $Y_i^{(N)}$

To each $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{Z}^N$ corresponds a monomial $\mathbf{z}^{\boldsymbol{\lambda}} := z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_N^{\lambda_N}$ of the total degree $|\boldsymbol{\lambda}| := \lambda_1 + \lambda_2 + \cdots + \lambda_N$. Let $\boldsymbol{\lambda}^+$ be the unique partition (we permit negative parts) obtained by ordering the elements of $\boldsymbol{\lambda}$: $\boldsymbol{\lambda}^+ = (\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq \lambda_N^+)$, $\lambda_{\sigma(i)}^+ = \lambda_i$ for a suitable permutation $\sigma := \{\sigma(1), \sigma(2), \ldots, \sigma(N)\}$ of the set $\{1, 2, \ldots, N\}$. We fix such a σ uniquely by requiring $\sigma(i) < \sigma(j)$ whenever $i < j$ and $\lambda_i = \lambda_j$.

There is a partial order relation in $\mathbb{Z}^N$. First, for two partitions $\boldsymbol{\lambda}^+ = (\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq \lambda_N^+)$ and $\boldsymbol{\mu}^+ = (\mu_1^+ \geq \mu_2^+ \geq \cdots \geq \mu_N^+)$ the partial order is defined by:

$$\boldsymbol{\lambda}^+ \succeq \boldsymbol{\mu}^+ \quad \Leftrightarrow \quad \sum_{j=1}^{i} \lambda_j^+ \geq \sum_{j=1}^{i} \mu_j^+ \quad (i = 1, 2, \ldots, N), \quad |\boldsymbol{\lambda}^+| = |\boldsymbol{\mu}^+|. \tag{2.11}$$

This order is extended to $\mathbb{Z}^N$ as follows. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{Z}^N$ put $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$ if:

$$\begin{aligned} &\text{either } \boldsymbol{\lambda}^+ \succ \boldsymbol{\mu}^+ \text{ , or } \boldsymbol{\lambda}^+ = \boldsymbol{\mu}^+ \text{ and} \\ &\text{the last non-zero element in } (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_N - \mu_N) \text{ is negative.} \end{aligned} \tag{2.12}$$

It is straightforward to verify that the action of $\xi_{i,j} := K_{i,j} g_{i,j}$ and the $Y_i^{(N)}$ in the monomial basis is triangular. More precisely:

$$\xi_{i,j} \mathbf{z}^{\boldsymbol{\lambda}} = \begin{cases} q^{-1} \mathbf{z}^{\boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda}^+ \succ \boldsymbol{\mu}^+} c(\boldsymbol{\lambda}, \boldsymbol{\mu}) \mathbf{z}^{\boldsymbol{\mu}} & (\lambda_i < \lambda_j), \\ q \mathbf{z}^{\boldsymbol{\lambda}} & (\lambda_i = \lambda_j), \\ q \mathbf{z}^{\boldsymbol{\lambda}} + (q - q^{-1}) \mathbf{z}^{(i,j)\boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda}^+ \succ \boldsymbol{\mu}^+} c(\boldsymbol{\lambda}, \boldsymbol{\mu}) \mathbf{z}^{\boldsymbol{\mu}} & (\lambda_i > \lambda_j) \end{cases} \tag{i}$$

where $(i,j)\boldsymbol{\lambda} := \boldsymbol{\lambda}|_{\lambda_i \leftrightarrow \lambda_j}$.

$$Y_i^{(N)} \mathbf{z}^{\boldsymbol{\lambda}} = p^{\lambda_i} q^{2\sigma(i) - N - 1} \mathbf{z}^{\boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda} \succ \boldsymbol{\mu}} c(\boldsymbol{\lambda}, \boldsymbol{\mu}) \mathbf{z}^{\boldsymbol{\mu}}. \tag{ii}$$

Let us put $\zeta_i(\boldsymbol{\lambda}) := p^{\lambda_i} q^{2\sigma(i) - N - 1}$ for $\boldsymbol{\lambda} \in \mathbb{Z}^N$ and $i = 1, 2, \ldots, N$. Since for generic p and q the equality $\zeta_i(\boldsymbol{\lambda}) = \zeta_i(\boldsymbol{\mu})$ $(i = 1, 2, \ldots, N)$ implies $\boldsymbol{\lambda} = \boldsymbol{\mu}$ we immediately come to the conclusion that the $Y_i^{(N)}$ admit a common eigenbasis $\{\Phi^{\boldsymbol{\lambda}}(\mathbf{z}) \mid \boldsymbol{\lambda} \in \mathbb{Z}^N\}$:

$$Y_i^{(N)} \Phi^{\boldsymbol{\lambda}}(\mathbf{z}) = \zeta_i(\boldsymbol{\lambda}) \Phi^{\boldsymbol{\lambda}}(\mathbf{z}), \quad (i = 1, 2, \ldots, N), \tag{2.13}$$

and

$$\Phi^{\boldsymbol{\lambda}}(\mathbf{z}) = \mathbf{z}^{\boldsymbol{\lambda}} + \sum_{\boldsymbol{\lambda} \succ \boldsymbol{\mu}} c(\boldsymbol{\lambda}, \boldsymbol{\mu}) \mathbf{z}^{\boldsymbol{\mu}}. \tag{2.14}$$

Following [7] we will refer to the Laurent polynomials $\Phi^{\boldsymbol{\lambda}}(\mathbf{z})$ as Non-symmetric Macdonald Polynomials (of type A).

The action of the finite Hecke algebra generators $g_{i,i+1}$ in the basis $\{\Phi^{\boldsymbol{\lambda}}(\mathbf{z}) \mid \boldsymbol{\lambda} \in \mathbb{Z}^N\}$ is summarized as follows [17]:

$$g_{i,i+1} \Phi^{\boldsymbol{\lambda}}(\mathbf{z}) = A_i(\boldsymbol{\lambda}) \Phi^{\boldsymbol{\lambda}}(\mathbf{z}) + B_i(\boldsymbol{\lambda}) \Phi^{(i,i+1)\boldsymbol{\lambda}}(\mathbf{z}), \tag{2.15}$$

where $(i, i+1)\boldsymbol{\lambda} := \boldsymbol{\lambda}|_{\lambda_i \leftrightarrow \lambda_{i+1}}$ and:

$$A_i(\boldsymbol{\lambda}) := \frac{(q - q^{-1})x}{x - 1}, \quad B_i(\boldsymbol{\lambda}) := \begin{cases} q^{-1}\{x\} & (\lambda_i > \lambda_{i+1}); \\ 0 & (\lambda_i = \lambda_{i+1}); \\ q^{-1} & (\lambda_i < \lambda_{i+1}), \end{cases} \tag{2.16}$$

$$\{x\} := \frac{(x - q^2)(q^2 x - 1)}{(x - 1)^2}, \quad x := \frac{\zeta_{i+1}(\boldsymbol{\lambda})}{\zeta_i(\boldsymbol{\lambda})}. \tag{2.17}$$

Note that when i is such that $\lambda_i = \lambda_{i+1}$ we have $\frac{\zeta_{i+1}(\boldsymbol{\lambda})}{\zeta_i(\boldsymbol{\lambda})} = q^2$ and hence $g_{i,i+1} \Phi^{\boldsymbol{\lambda}}(\mathbf{z}) = q \Phi^{\boldsymbol{\lambda}}(\mathbf{z})$.

2.3 Finite q-wedge Product

Let $n \geq 2$ and N be positive integers. We set: $V := \mathbb{C}^n$ with a base $\{v_1, v_2, \dots, v_n\}$ and $V(z) := \mathbb{C}[z^{\pm 1}] \otimes V$ with a base $\{z^m v_\epsilon\}$, $m \in \mathbb{Z}$, $\epsilon \in \{1, 2, \dots, n\}$. Often it will be convenient to set $k = \epsilon - nm$, $u_k := z^m v_\epsilon$. Then $\{u_k \mid k \in \mathbb{Z}\}$ is a base in $V(z)$. In what follows we will use both notations: u_k and $z^m v_\epsilon$ switching between them without further alert. The q-wedge product of spaces $V(z)$ is defined as a suitable quotient of the tensor product $\otimes^N V(z) \cong \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}] \otimes (\otimes^N V)$. To describe this quotient introduce an action of the finite Hecke algebra in $\otimes^N V$:

$$\begin{aligned} &T_i = S_{i,i+1}, \quad (i = 1, 2, \dots, N-1) \\ &S = -q^{-1} \sum_{1 \leq \epsilon \leq n} E^{\epsilon,\epsilon} \otimes E^{\epsilon,\epsilon} + (q - q^{-1}) \sum_{1 \leq \epsilon < \epsilon' \leq n} E^{\epsilon,\epsilon} \otimes E^{\epsilon',\epsilon'} - \sum_{1 \leq \epsilon \neq \epsilon' \leq n} E^{\epsilon,\epsilon'} \otimes E^{\epsilon',\epsilon}, \end{aligned} \tag{2.18}$$

where $E^{\epsilon',\epsilon} \in End(V)$ is specified by $E^{\epsilon',\epsilon} v_\alpha = \delta_{\epsilon,\alpha} v_{\epsilon'}$ and $S_{i,i+1}$ signifies S acting in the i-th and $i+1$-th factors in $\otimes^N V$.

Remark The Hecke generators T_i that are used in [1] are related to the generators which we use in this paper as follows:

$$T_i = q K_{i,i+1}(g_{i,i+1} + S^{-1}_{i,i+1}) - I. \tag{2.19}$$

Now define $\Omega (\subset \otimes^N V(z))$ as:

$$\Omega = \sum_{i=1}^{N-1} Ker(g_{i,i+1} + S^{-1}_{i,i+1}). \tag{2.20}$$

In this setting the q-wedge product $\wedge^N V(z)$ is defined as the quotient:

$$\wedge^N V(z) = \otimes^N V(z)/\Omega. \tag{2.21}$$

This definition is equivalent to the definition in [1] due to the Remark above. Notice that for $q = 1$ the q-wedge product is just the usual exterior (wedge) product of the spaces $V(z)$. In what follows we will use the term "wedge product" always for the q-deformed wedge product (2.21).

Let $\Lambda : \otimes^N V(z) \to \wedge^N V(z)$ be the quotient map specified by (2.21). The image of a pure tensor $u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}$ under this map is called a wedge and is denoted by:

$$u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_N} := \Lambda(u_{k_1} \otimes u_{k_2} \otimes \cdots \otimes u_{k_N}). \tag{2.22}$$

In [1] it is proven that a basis in $\wedge^N V(z)$ is formed by the normally ordered wedges, that is the wedges (2.22) such that $k_1 > k_2 > \cdots > k_N$. Any wedge can be written as a linear combination of the normally ordered wedges by using the normal ordering rules [1]:

$$u_l \wedge u_m = -u_m \wedge u_l, \quad \text{for } l = m \bmod n, \tag{2.23}$$

$$\begin{aligned} u_l \wedge u_m = &-q u_m \wedge u_l + (q^2 - 1)(u_{m-i} \wedge u_{l+i} - q u_{m-n} \wedge u_{l+n} + \\ &+ q^2 u_{m-n-i} \wedge u_{l+n+i} + \dots), \\ &\text{for } l < m, m - l = i \bmod n, 0 < i < n. \end{aligned} \tag{2.24}$$

The sum above continues as long as the wedges in the right-hand side are normally ordered.

2.4 Actions of $U'_q(\widehat{\mathfrak{sl}}_n)$ in the wedge product.

With each of the two $\widehat{H}_N(q)$ actions (2.7,2.9) we associate an action of the algebra $U'_q(\widehat{\mathfrak{sl}}_n)$ in the tensor product $\otimes^N V(z)$. Let $E_i^{\epsilon,\epsilon'}$ denote the operator which acts trivially in all factors in $\mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}] \otimes (\otimes^N \mathbb{C}^n)$ except the i-th factor in $\otimes^N \mathbb{C}^n$ where it acts as $E^{\epsilon,\epsilon'}$. The action of the generators $\{E_\epsilon, F_\epsilon, K_\epsilon\}$, $(\epsilon \in$

$\{0,1,\ldots,n-1\}$) of $U_q'(\widehat{\mathfrak{sl}}_n)$ (our conventions on $U_q'(\widehat{\mathfrak{sl}}_n)$ are summarized in the Appendix) then is defined as follows:

$$K_\epsilon = K_1^\epsilon K_2^\epsilon \ldots K_N^\epsilon, \quad K_i^\epsilon = q^{E_i^{\epsilon,\epsilon}-E_i^{\epsilon+1,\epsilon+1}}, \tag{2.25}$$

$$E_\epsilon = \sum_{i=1}^{N} y_i^{-\delta_{0,\epsilon}} E_i^{\epsilon,\epsilon+1} K_{i+1}^\epsilon \ldots K_N^\epsilon, \tag{2.26}$$

$$F_\epsilon = \sum_{i=1}^{N} y_i^{\delta_{0,\epsilon}} (K_1^\epsilon)^{-1} \ldots (K_{i-1}^\epsilon)^{-1} E_i^{\epsilon+1,\epsilon}, \quad \epsilon = 0,\ldots,n-1\ , \tag{2.27}$$

where in the right-hand side we regard the indices ϵ,$\epsilon+1$ modulo n.

The substitution $y_i = z_i^{-1}$ in these expressions gives the $U_q'(\widehat{\mathfrak{sl}}_n)$ action which was considered in [1] – denote this action by $U_1^{(N)}$. The other choice of the affine Hecke algebra generators: $y_i = q^{1-N}Y_i^{(N)}$ gives another action of $U_q'(\widehat{\mathfrak{sl}}_n)$ in $\otimes^N V(z)$ – this action the principal object of study in the present paper , we denote it by $U_0^{(N)}$.

The centre of $\widehat{H}_N(q)$ is generated by symmetric polynomials in $y_i^{\pm 1}$ so we consider two abelian algebras: $H_1^{(N)}$ with generators [1]:

$$B_a^{(N)} := z_1^a + z_2^a + \cdots + z_N^a, \quad (a \in \{\pm 1, \pm 2, \ldots, \}); \tag{2.28}$$

And $H_0^{(N)}$ with generators [3]:

$$h_a^{(N)} := (q^{1-N}Y_1^{(N)})^a + (q^{1-N}Y_2^{(N)})^a + \cdots + (q^{1-N}Y_N^{(N)})^a, \quad (a \in \{\pm 1, \pm 2, \ldots, \}). \tag{2.29}$$

Obviously $H_j^{(N)}$ commutes with $U_j^{(N)}$ for $j=0,1$.

It is straightforward to verify by using the relations of the affine Hecke algebra, and the explicit form of the operator S (2.18) that the actions $U_j^{(N)}$,$H_j^{(N)}$ for $j=0,1$ preserve the subspace Ω.

This implies that $U_j^{(N)}$, $H_j^{(N)}$; $(j=0,1)$ are well-defined in the wedge product $\wedge^N V(z)$, and from now on we consider these actions as defined in $\wedge^N V(z)$.

2.5 Semi-infinite wedge product

For $M \in \mathbb{Z}$ the space of semi-infinite wedges F_M is defined in [1] as the linear span of semi-infinite monomials:

$$u_{k_1} \wedge u_{k_2} \wedge u_{k_3} \wedge \quad \ldots \quad , \tag{2.30}$$

such that for $i >> 1$ the asymptotic condition $k_i = M - i + 1$ holds. The vacuum semi-infinite monomial in F_M is specified by $k_i = M - i + 1$, $i = 1,2,\ldots$ and is denoted by $|M\rangle$:

$$|M\rangle = u_M \wedge u_{M-1} \wedge u_{M-2} \wedge \quad \ldots \quad . \tag{2.31}$$

The normal ordering rules (2.23, 2.24) imply that the normally ordered semi-infinite monomials – that is (2.30) with $k_1 > k_2 > k_3 > \ldots$ form a basis in F_M.

The level-0 action $U_1^{(N)}$ in the limit $N \to \infty$ was used in [1] to define a *level-1* action of $U_q'(\widehat{\mathfrak{sl}}_n)$ in the space F_M, such that as an $U_q'(\widehat{\mathfrak{sl}}_n)$-module F_M is isomorphic to the Fock space module introduced in [2]. The abelian algebra $H_1^{(N)}$ in the same limit gives rise to an action of the *Heisenberg algebra* in F_M.

The two main problems which we address in the present paper are: 1) To define a *level-0* $U_q'(\widehat{\mathfrak{sl}}_n)$-action in F_M starting from the action $U_0^{(N)}$ in $\wedge^N V(z)$. 2) To construct the irreducible decomposition of the Fock space F_M with respect to this action.

3 Decomposition of the finite wedge product

In this section we find the decomposition of the wedge product $\wedge^N V(z)$ with respect to the $U_q'(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(N)}$. In order to derive this decomposition we construct a suitable base of $\wedge^N V(z)$ by using the Non-symmetric Macdonald Polynomials.

3.1 A base in $\wedge^N V(z)$

Let $\mathbf{e} := (\epsilon_1, \epsilon_2, \ldots, \epsilon_N)$ where $\epsilon_i \in \{1, 2, \ldots, n\}$. For a sequence $\mathbf{e}$ we set

$$\mathbf{v_e} := v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_N} \quad (\in \otimes^N \mathbb{C}^n). \tag{3.1}$$

A sequence $\mathbf{m} := (m_1, m_2, \ldots, m_N)$ from $\mathbb{Z}^N$ is called n-strict if it contains no more than n equal elements of any given value. Let us define the sets $\mathcal{M}_N^n$ and $\mathcal{E}(\mathbf{m})$ by

$$\mathcal{M}_N^n := \{\mathbf{m} = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N \mid m_1 \leq m_2 \leq \cdots \leq m_N,\ \mathbf{m} \text{ is } n\text{-strict } \}, \tag{3.2}$$

and for $\mathbf{m} \in \mathcal{M}_N^n$

$$\mathcal{E}(\mathbf{m}) := \{\mathbf{e} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_N) \mid \epsilon_i > \epsilon_{i+1} \text{ for all } i \text{ s.t. } m_i = m_{i+1} \ \}. \tag{3.3}$$

In these notations the set

$$\{w(\mathbf{m}, \mathbf{e}) := \Lambda(\mathbf{z}^{\mathbf{m}} \otimes \mathbf{v_e}) \equiv z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \cdots \wedge z^{m_N} v_{\epsilon_N} \quad | \quad \mathbf{m} \in \mathcal{M}_N^n, \mathbf{e} \in \mathcal{E}(\mathbf{m})\}. \tag{3.4}$$

is nothing but the base of the normally ordered wedges in $\wedge^N V(z)$:

$$\{w(\mathbf{m}, \mathbf{e}) \quad | \quad \mathbf{m} \in \mathcal{M}_N^n, \mathbf{e} \in \mathcal{E}(\mathbf{m})\} = \{u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_N} \quad | \quad k_1 > k_2 > \cdots > k_N\},$$
$$k_i = \epsilon_i - n m_i.$$

For the purpose of the $U_q'(\widehat{\mathfrak{sl}}_n)$-decomposition we construct another base. The elements of this new base have the same labels $\mathbf{m} \in \mathcal{M}_N^n, \mathbf{e} \in \mathcal{E}(\mathbf{m})$ as the elements of the base of the normally ordered wedges.

For $\mathbf{m} \in \mathcal{M}_N^n, \mathbf{e} \in \mathcal{E}(\mathbf{m})$ let us put

$$\phi(\mathbf{m}, \mathbf{e}) := \Lambda(\Phi^{\mathbf{m}}(\mathbf{z}) \otimes \mathbf{v_e}). \tag{3.6}$$

Notice that at $q = 1$ we have $\phi(\mathbf{m}, \mathbf{e}) \equiv w(\mathbf{m}, \mathbf{e})$.

Proposition 1 *The set $\{\phi(\mathbf{m}, \mathbf{e}) \mid \mathbf{m} \in \mathcal{M}_N^n, \mathbf{e} \in \mathcal{E}(\mathbf{m})\}$ is a base of $\wedge^N V(z)$.*

Proof. To show that $\wedge^N V(z) = span_{\mathbb{C}}\{\phi(\mathbf{m}, \mathbf{e}) \mid \mathbf{m} \in \mathcal{M}_N^n, \mathbf{e} \in \mathcal{E}(\mathbf{m})\}$ we use the formulas (2.15) for the action of the Hecke algebra generators on Non-symmetric Macdonald Polynomials together with the fact that for any $i \in \{1, 2, \ldots, N-1\}$ we have

$$Im(g_{i,i+1} - S_{i,i+1}) \subset Ker(g_{i,i+1} + S_{i,i+1}^{-1}) \Rightarrow \Lambda((g_{i,i+1} - S_{i,i+1})f) = 0 \quad \forall f \in \otimes^N V(z). \tag{3.7}$$

First, the eq. (2.15) and (3.7) allow us to write (at generic p and q):

$$\wedge^N V(z) = span_{\mathbb{C}}\{\Lambda(\Phi^{\boldsymbol{\lambda}}(\mathbf{z}) \otimes \mathbf{v}) \mid \boldsymbol{\lambda} \in \mathbb{Z}^N, \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N; \mathbf{v} \in \otimes^N \mathbb{C}^n\}. \tag{3.8}$$

Next, we observe that (2.15,3.7) imply:

$$\Lambda(\Phi^{\boldsymbol{\lambda}}(\mathbf{z}) \otimes (q + S_{i,i+1}^{-1})\mathbf{v}) = 0 \quad \text{whenever } \lambda_i = \lambda_{i+1}. \tag{3.9}$$

From the last equation it follows that a vector $\Lambda(\Phi^{\boldsymbol{\lambda}}(\mathbf{z}) \otimes \mathbf{v})$ is equal to zero if the sequence $\boldsymbol{\lambda}$ is not n-strict. Now using (3.9) together with the explicit form of the operators $S_{i,i+1}$ (2.18) we derive from (3.8) that the vectors $\phi(\mathbf{m}, \mathbf{e}) := \Lambda(\Phi^{\mathbf{m}}(\mathbf{z}) \otimes \mathbf{v_e})$ with $\mathbf{m} \in \mathcal{M}_N^n$ and $\mathbf{e} \in \mathcal{E}(\mathbf{m})$ span the wedge product.

To demonstrate linear independence of these vectors we consider the limit $q=1$ in which limit the $\phi(\mathbf{m},\mathbf{e})$ coincide with the $w(\mathbf{m},\mathbf{e})$ – elements of the base of the normally ordered wedges. ■

Since $\Phi^{\mathbf{m}}(\mathbf{z})$ is an eigenvector of the operators $Y_i^{(N)}$ it is clear, that $U_0^{(N)}$ and $H_0^{(N)}$ preserve the subspace

$$E^{\mathbf{m}} := \oplus_{\mathbf{e}\in\mathcal{E}(\mathbf{m})}\mathbb{C}\phi(\mathbf{m},\mathbf{e}) \tag{3.10}$$

for any $\mathbf{m}\in\mathcal{M}_N^n$. Moreover it is easy to see, that the $E^{\mathbf{m}}$ is an eigenspace of the operators $h_a^{(N)}$ which generate $H_0^{(N)}$. The eigenvalue $h_a^{(N)}(\mathbf{m})$ of $h_a^{(N)}$ $(a=\pm1,\pm2,\dots)$ for this eigenspace is

$$h_a^{(N)}(\mathbf{m}) = \sum_{i=1}^{N}(q^{1-N}\zeta_i(\mathbf{m}))^a = \sum_{i=1}^{N} p^{am_i}q^{2a(1-i)}. \tag{3.11}$$

Note also, that the $E^{\mathbf{m}}$ is an eigenspace of the degree operator $z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+\cdots+z_N\frac{\partial}{\partial z_N}$ with the eigenvalue $|\mathbf{m}| := m_1+m_2+\cdots+m_N$.

In the rest of this section we will describe the structure of the $E^{\mathbf{m}}$ for a fixed $\mathbf{m}\in\mathcal{M}_N^n$ as an $U_q'(\widehat{\mathfrak{sl}}_n)$-module with the action $U_0^{(N)}$.

3.2 The $U_q'(\widehat{\mathfrak{sl}}_n)$-module $E^{\mathbf{m}}$

For $a_1,a_2,\dots,a_r\in\mathbb{C}$ let $\pi^{(r)}_{a_1,\dots,a_r}$ be the evaluation action of $U_q'(\widehat{\mathfrak{sl}}_n)$ defined in $\otimes^N\mathbb{C}^n$ by

$$\pi^{(r)}_{a_1,\dots,a_r}(K_\epsilon) = K_1^\epsilon K_2^\epsilon\dots K_r^\epsilon,\quad K_i^\epsilon = q^{E_i^{\epsilon,\epsilon}-E_i^{\epsilon+1,\epsilon+1}}, \tag{3.12}$$

$$\pi^{(r)}_{a_1,\dots,a_r}(E_\epsilon) = \sum_{i=1}^{r} a_i^{\delta_{0,\epsilon}}E_i^{\epsilon,\epsilon+1}K_{i+1}^\epsilon\dots K_r^\epsilon, \tag{3.13}$$

$$\pi^{(r)}_{a_1,\dots,a_r}(F_\epsilon) = \sum_{i=1}^{r} a_i^{-\delta_{0,\epsilon}}(K_1^\epsilon)^{-1}\dots(K_{i-1}^\epsilon)^{-1}E_i^{\epsilon+1,\epsilon},\quad \epsilon=0,\dots,n-1\ , \tag{3.14}$$

where in the right-hand side we regard the indices ϵ,$\epsilon+1$ modulo n.

In this notation we set for $\mathbf{m}\in\mathcal{M}_N^n$:

$$\pi(\mathbf{m}) := \pi^{(N)}_{a_1,\dots,a_N}\quad \text{wherein we put } a_i = q^{N-1}\zeta_i(\mathbf{m})^{-1}. \tag{3.15}$$

Since $\zeta_{i+1}(\mathbf{m}) = q^2\zeta_i(\mathbf{m})$ whenever $m_i=m_{i+1}$ the action $\pi(\mathbf{m})$ has the following property:

$$(q+S_{i,i+1}^{-1})\pi(\mathbf{m}) = \pi(\mathbf{m})|_{\zeta_{i+1}(\mathbf{m})\leftrightarrow\zeta_i(\mathbf{m})}(q+S_{i,i+1}^{-1})\quad \text{for all } i \text{ s.t. } m_i=m_{i+1}\ . \tag{3.16}$$

As the result the $\pi(\mathbf{m})$ is well-defined in the space:

$$W^{\mathbf{m}} := \otimes^N\mathbb{C}^n/\sum_{\{i|m_i=m_{i+1}\}} Ker(q+S_{i,i+1}^{-1}). \tag{3.17}$$

Let $\iota_{\mathbf{m}}$: $\otimes^N\mathbb{C}^n\to W^{\mathbf{m}}$ be the quotient map defined by (3.17). We define a map

$$\beta: E^{\mathbf{m}} = span_{\mathbb{C}}\{\Lambda(\Phi^{\mathbf{m}}(\mathbf{z})\otimes v)\mid v\in\otimes^N\mathbb{C}^n\}(=\oplus_{\mathbf{e}\in\mathcal{E}(\mathbf{m})}\mathbb{C}\Lambda(\Phi^{\mathbf{m}}(\mathbf{z})\otimes\mathbf{v_e}))\ \to\ W^{\mathbf{m}} \tag{3.18}$$

by setting $\beta(\Lambda(\Phi^{\mathbf{m}}(\mathbf{z})\otimes v)) = \iota_{\mathbf{m}}(v)$.

Proposition 2 *The map β (3.18) is an isomorphism of the $U_q'(\widehat{\mathfrak{sl}}_n)$-modules $(E^{\mathbf{m}},\ U_0^{(N)})$ and $(W^{\mathbf{m}},\ \pi(\mathbf{m}))$.*

Proof. To show that the β is an isomorphism of the linear spaces it is sufficient to observe that the set $\{\iota_{\mathbf{m}}(\mathbf{v_e})\mid \mathbf{e}\in\mathcal{E}(\mathbf{m})\}$ is a base in $W^{\mathbf{m}}$.

The map β is an intertwiner of the actions $U_0^{(N)}$ and $\pi(\mathbf{m})$ since for any generator x of $U_0^{(N)}$ we have:

$$x.\Lambda(\Phi^{\mathbf{m}}(\mathbf{z})\otimes v) = \Lambda(\Phi^{\mathbf{m}}(\mathbf{z})\otimes\pi(\mathbf{m})(x).v). \tag{3.19}$$

■

Now let us subdivide the sequence $\mathbf{m} = (m_1 \leq m_2 \leq \cdots \leq m_N)$ into subsequences which comprise equal elements:

$$\mathbf{m} = (m_1 = \cdots = m_{r_1} < m_{1+r_1} = \cdots = m_{r_2} < \cdots < m_{1+r_J} = \cdots = m_N). \tag{3.20}$$

Notice that since $\mathbf{m}$ is n-strict we have for $1 \leq k \leq J+1$ the inequalities $1 \leq r_k - r_{k-1} \leq n$ (here we put $r_0 := 0$ and $r_{J+1} := N$). As the $U'_q(\widehat{\mathfrak{sl}}_n)$-module the space $W^{\mathbf{m}}$ then is represented as the following tensor product:

$$W^{\mathbf{m}} = V[p^{-m_{r_1}} q^{2(r_1-1)}, r_1] \otimes V[p^{-m_{r_2}} q^{2(r_2-1)}, r_2 - r_1] \otimes \cdots \otimes V[p^{-m_N} q^{2(N-1)}, N - r_J] \tag{3.21}$$

where $V[a,j]$ $(1 \leq j \leq n)$ is the $U'_q(\widehat{\mathfrak{sl}}_n)$-module with the action $\pi^{(j)}_{a_1,\ldots,a_j}$ such that $a_1 := a$, $a_2 := q^{-2}a$, $\ldots$, $a_j := q^{-2(j-1)}a$ (3.12 - 3.14), and as a linear space

$$V[a,j] = \otimes^j \mathbb{C}^n / \sum_{i=1}^{j-1} Ker(q + S^{-1}_{i,i+1}). \tag{3.22}$$

Our next task is to describe the $V[a,j]$ $(1 \leq j \leq n)$. First of all for $1 \leq j \leq n-1$ as $U_q(\mathfrak{sl}_n)$-module ($U_q(\mathfrak{sl}_n) \subset U'_q(\widehat{\mathfrak{sl}}_n)$) the $V[a,j]$ is irreducible and isomorphic to the highest weight module $V(\Lambda_j)$ with the fundamental $U_q(\mathfrak{sl}_n)$ highest weight Λ_j. When $j = n$; the $V[a,j]$ is the 1-dimensional trivial representation of $U_q(\mathfrak{sl}_n)$. Thus $V[a,j]$ $(1 \leq j \leq n)$ is an irreducible $U'_q(\widehat{\mathfrak{sl}}_n)$-module, and in order to give the complete specification of $V[a,j]$ for $1 \leq j \leq n-1$ it is sufficient to describe associated Drinfel'd Polynomials. In the conventions of [6] which we recall in the Appendix these are provided by the following Lemma:

Lemma 1 *For $1 \leq j \leq n-1$ the $V[a,j]$ is an irreducible $U'_q(\widehat{\mathfrak{sl}}_n)$-module with the Drinfel'd Polynomials*

$$P_k(u) = \begin{cases} u - q^{j-2}a^{-1} & \text{for } k = j, \\ 1 & \text{for } 1 \leq k \leq n-1,\ k \neq j. \end{cases} \tag{3.23}$$

Using the results of [8] we can claim, that for generic q and p the representation $W^{\mathbf{m}}$ (3.21) is irreducible, therefore the Drinfel'd Polynomials of $W^{\mathbf{m}}$ are just products of the Drinfel'd Polynomials associated with the factors $V[p^{-m_{r_k}} q^{2(r_k - r_{k-1})}, r_k - r_{k-1}]$ (Cf.[6]). This leads to the main Proposition of this section:

Proposition 3 *For generic p and q the $E^{\mathbf{m}} \cong W^{\mathbf{m}}$ is irreducible and with notations of* (3.20) the Drinfel'd Polynomials of $E^{\mathbf{m}} \cong W^{\mathbf{m}}$ are

$$P_i(u) = \prod_{\{1 \leq k \leq J+1 \mid r_k - r_{k-1} = i\}} (u - p^{m_{r_k}} q^{-r_k - r_{k-1}}) \qquad (i \in \{1, 2, \ldots, n-1\}). \tag{3.24}$$

The proofs of this Proposition and the Lemma 1 are discussed in the Appendix.

Finally we note that

$$\wedge^N V(z) = \bigoplus_{\mathbf{m} \in \mathcal{M}^n_N} E^{\mathbf{m}}. \tag{3.25}$$

This is the desired decomposition of the finite wedge product with respect to the $U'_q(\widehat{\mathfrak{sl}}_n)$ action U_0.

4 A level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ in the Fock space

In this section we will define a level-0 action of the $U_q'(\widehat{\mathfrak{sl}}_n)$ in the space of semi-infinite wedges F_M (or equivalently in the level-1 Fock space module of the same algebra – $U_q'(\widehat{\mathfrak{sl}}_n)$ – see subsection 2.5) starting from the action $U_0^{(N)}$ which was defined in sec. 2 in the finite wedge product. The level-0 action in the space F_M is constructed by taking a suitable projective limit of $U_0^{(N)}$ and can be thought of as an appropriate, well-defined limit of $U_0^{(N)}$ when the number of particles N goes to infinity.

In this section we allow the parameter p to be arbitrary complex number, whrereas the q is still required to be generic.

Let $w = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_N} \equiv z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \cdots \wedge z^{m_N} v_{\epsilon_N}$, $w \in \wedge^N V(z)$ be a normally ordered wedge: $k_1 > k_2 > \cdots > k_N$. Often it will be convenient to label this wedge by the two sequences $\mathbf{m} \in \mathcal{M}_N^n$ and $\mathbf{e} \in \mathcal{E}(\mathbf{m})$:

$$\begin{aligned} \mathbf{m} &= (m_1 \leq m_2 \leq \cdots \leq m_N), \quad m_i \in \mathbb{Z}, && (4.1) \\ \mathbf{e} &= (\epsilon_1, \epsilon_2, \ldots, \epsilon_N), \quad 1 \leq \epsilon_i \leq n, && (4.2) \\ & \text{such that } k_i = \epsilon_i - n m_i, && (4.3) \end{aligned}$$

and write: $w := w(\mathbf{m}, \mathbf{e})$. From now on we will use the notation $w(\mathbf{m}, \mathbf{e})$ *exclusively* for normally ordered wedges.

Recall that by the definitions (3.2, 3.3) of the $\mathcal{M}_N^n$ and $\mathcal{E}(\mathbf{m})$ the $\mathbf{m}$ in (4.1) is always n-strict sequence and that $\epsilon_i > \epsilon_{i+1}$ whenever $m_i = m_{i+1}$.

Throughout this section we fix an integer M and $0 \leq s \leq n-1$ such that $M = s \bmod n$. Let $r \in \{0, 1, 2, \ldots\}$; and let $w(\mathbf{m}^0, \mathbf{e}^0) \in \wedge^{s+nr} V(z)$ be defined as:

$$w(\mathbf{m}^0, \mathbf{e}^0) := u_M \wedge u_{M-1} \wedge \cdots \wedge u_{M-(s+nr)+1}. \tag{4.4}$$

Here the sequences $\mathbf{m}^0$ and $\mathbf{e}^0$ are as follows:

$$\begin{aligned} \mathbf{m}^0 = (m_1^0, m_2^0, \ldots, m_{s+nr}^0) &:= (\underbrace{m^0, \ldots, m^0}_{s}, \underbrace{m^0+1, \ldots, m^0+1}_{n}, \\ & \quad \underbrace{m^0+2, \ldots, m^0+2}_{n}, \ldots, \underbrace{m^0+r, \ldots, m^0+r}_{n}) && (4.5) \\ \mathbf{e}^0 = (\epsilon_1^0, \epsilon_2^0, \ldots, \epsilon_{s+nr}^0) &:= (\underbrace{s, s-1, \ldots, 1}_{s}, \underbrace{n, n-1, \ldots, 1}_{n}, \\ & \quad \underbrace{n, n-1, \ldots, 1}_{n}, \ldots, \underbrace{n, n-1, \ldots, 1}_{n}), && (4.6) \\ & \quad \text{where} \quad m^0 := \frac{s-M}{n} \qquad (0 \leq s \leq n-1). && (4.7) \end{aligned}$$

We will call these two sequences *vacuum* sequences, and the $w(\mathbf{m}^0, \mathbf{e}^0)$ – *vacuum vector* of $\wedge^{s+nr} V(z)$.

Define $V_M^{s+nr} \subset \wedge^{s+nr} V(z)$ as:

$$V_M^{s+nr} = \bigoplus_{\substack{\mathbf{m} \in \mathcal{M}_N^n, \mathbf{e} \in \mathcal{E}(\mathbf{m}) \\ m_{s+nr} \leq m_{s+nr}^0}} \mathbb{C} w(\mathbf{m}, \mathbf{e}). \tag{4.8}$$

Notice that the condition $m_{s+nr} \leq m_{s+nr}^0$ in this definition is equivalent to the condition:

$$m_i \leq m_i^0 \quad \text{for all } i = 1, 2, \ldots, s+nr \tag{4.9}$$

because the sequence $\mathbf{m}$ is n-strict and non-decreasing.

Proposition 4 *The $U_q'(\widehat{\mathfrak{sl}}_n)$ action $U_0^{(s+nr)}$ and the operators $h_l^{(s+nr)}$ ($l \in \mathbb{Z}_{\neq 0}$) preserve the subspace V_M^{s+nr}.*

To prove this and other Propositions we need two Lemmas. The first of these Lemmas concerns properties of the operators (2.10) and operators:

$$\xi_{i,j} = K_{i,j} g_{i,j}, \tag{4.10}$$

where $K_{i,j}$ and $g_{i,j}$ are defined in (2.8).

Lemma 2 *Let $\mathbf{z}^{\mathbf{n}} \equiv z_1^{n_1} z_2^{n_2} \dots z_N^{n_N}$ be a monomial in $\mathbb{C}[z_1^{\pm}, \dots, z_N^{\pm}]$, and let $a = \max\{n_1, \dots, n_N\}$.*

Then

$$\begin{aligned} \xi_{i,j}^{\pm 1} \mathbf{z}^{\mathbf{n}} &= \sum_{\mathbf{n}'} c_{\pm}(\mathbf{n}, \mathbf{n}') \mathbf{z}^{\mathbf{n}'}, && (4.11)\\ (Y_i^{(N)})^k \mathbf{z}^{\mathbf{n}} &= \sum_{\mathbf{n}'} c_k(\mathbf{n}, \mathbf{n}') \mathbf{z}^{\mathbf{n}'}, \quad k = \pm 1, \pm 2, \dots \quad . && (4.12) \end{aligned}$$

where $c_{\pm}(\mathbf{n}, \mathbf{n}'), c_k(\mathbf{n}, \mathbf{n}')$ are coefficients, and the summation ranges over $\mathbf{n}'$ such that:

$$\begin{aligned} n_1', n_2', \dots, n_N' &\leq a, && (4.13)\\ \#\{n_i' | n_i' = a\} &\leq \#\{n_i | n_i = a\}, && (4.14)\\ n_1' + n_2' + \dots + n_N' &= n_1 + n_2 + \dots + n_N. && (4.15) \end{aligned}$$

Proof. To prove the statement about the summation range (4.13 - 4.15) in (4.11) we use the explicit formulas for the action of $\xi_{i,j}$ and $\xi_{i,j}^{-1}$ on monomials:

$$\begin{aligned} &\text{for} \quad n_i < n_j \quad : && (4.16)\\ \xi_{i,j}^{\pm 1} z_i^{n_i} z_j^{n_j} &= q^{\mp 1} z_i^{n_i} z_j^{n_j} \mp (q - q^{-1}) \sum_{k=1}^{n_j - n_i - 1} z_i^{n_i + k} z_j^{n_j - k}, \\ &\text{for} \quad n_i = n_j \quad : && (4.17)\\ \xi_{i,j}^{\pm 1} z_i^{n_i} z_j^{n_j} &= q^{\pm 1} z_i^{n_i} z_j^{n_j}, \\ &\text{for} \quad n_i > n_j \quad : && (4.18)\\ \xi_{i,j}^{\pm 1} z_i^{n_i} z_j^{n_j} &= q^{\pm 1} z_i^{n_i} z_j^{n_j} \pm (q - q^{-1}) z_i^{n_j} z_j^{n_i} \pm (q - q^{-1}) \sum_{k=1}^{n_i - n_j - 1} z_i^{n_i - k} z_j^{n_j + k}. \end{aligned}$$

The statements (4.11, 4.13 - 4.15) immediately follow from these formulas. The statements (4.12, 4.13 - 4.15) follow from (4.11, 4.13 - 4.15) and (2.10). ■

We will also need the following Lemma which shows triangularity of the normal ordering.

Lemma 3 *Let $v \in \otimes^N \mathbb{C}^n$, and Λ be the quotient map defined by (2.21).*

Then in the notations of Lemma 2 the following holds:

$$\Lambda(\mathbf{z}^{\mathbf{n}} \otimes v) = \sum_{\mathbf{n}', \mathbf{e}} c(\mathbf{n}, v; \mathbf{n}', \mathbf{e}) w(\mathbf{n}', \mathbf{e}), \tag{4.19}$$

where $c(\mathbf{n}, v; \mathbf{n}', \mathbf{e})$ is a coefficient and the summation ranges over $\mathbf{n}'$ such that $(\mathbf{n}')^+ \preceq (\mathbf{n})^+$ and consequently:

$$\begin{aligned} n_1', n_2', \dots, n_N' &\leq a (:= \max\{n_1, \dots, n_N\}), && (4.20)\\ \#\{n_i' | n_i' = a\} &\leq \#\{n_i | n_i = a\}, && (4.21)\\ n_1' + n_2' + \dots + n_N' &= n_1 + n_2 + \dots + n_N && (4.22) \end{aligned}$$

Proof. Use the normal ordering rules (2.23, 2.24). ■

Proof of the Proposition 4 Let $w(\mathbf{m},\mathbf{e}) \in V_M^{s+nr}$ – that is $m_{s+nr} \leq m^0_{s+nr}$. We prove the Proposition by considering the action of the generators of $U_0^{(s+nr)}$ on $w(\mathbf{m},\mathbf{e})$.

Action of any of the generators (2.25 - 2.27) of the $U_q(\mathfrak{sl}_n)$ subalgebra of $U_0^{(s+nr)}$ on this wedge results in a linear combination of wedges $w(\mathbf{m},\mathbf{e}')$ with the same sequence $\mathbf{m}$ as in $w(\mathbf{m},\mathbf{e})$. Therefore the action of the $U_q(\mathfrak{sl}_n)$ subalgebra preserves V_M^{s+nr}. By the same token K_0 (2.25) preserves V_M^{s+nr} as well.

Consider now the vector:

$$F_0.w(\mathbf{m},\mathbf{e}) = \sum_{i=1}^{N} \Lambda(q^{1-N} Y_i^{(N)}.\mathbf{z}^{\mathbf{m}} \otimes (K_1^0)^{-1} \dots (K_{i-1}^0)^{-1} E_i^{1,n}.\mathbf{v}_{\mathbf{e}}), \tag{4.23}$$

where we put $N := s + nr$ and $\mathbf{v}_{\mathbf{e}} := v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \cdots \otimes v_{\epsilon_N} \in \otimes^N \mathbb{C}^n$. In each of the summands apply first Lemma 2 to express $Y_i^{(N)}.\mathbf{z}^{\mathbf{m}}$ as a linear combination of monomials, then apply Lemma 4 to express the result as a linear combination of the normally ordered wedges:

$$F_0.w(\mathbf{m},\mathbf{e}) = \sum_{\mathbf{m}',\mathbf{e}'} c(\mathbf{m},\mathbf{e};\mathbf{m}',\mathbf{e}') w(\mathbf{m}',\mathbf{e}'). \tag{4.24}$$

Due to (4.13) in Lemma 2 and (4.20) in Lemma 4 in the last formula we have: $m'_{s+nr} \leq m_{s+nr} \leq m^0_{s+nr}$ and thus $F_0.w(\mathbf{m},\mathbf{e}) \in V_M^{s+nr}$ by the definition (4.8) of the V_M^{s+nr} .

For the generator E_0 and the operators $h_l^{(s+nr)}$ the proof is done by the same arguments as for F_0. ■

Introduce the degree $|w(\mathbf{m},\mathbf{e})|$ of a wedge $w(\mathbf{m},\mathbf{e}) \in V_M^{s+nr}$ by:

$$|w(\mathbf{m},\mathbf{e})| = \sum_{i=1}^{s+nr} m_i^0 - m_i, \tag{4.25}$$

where the vacuum sequence $\mathbf{m}^0$ is defined in (4.5). We have:

$$V_M^{s+nr} = \bigoplus_{k\geq 0} V_M^{s+nr,k}, \tag{4.26}$$

$$V_M^{s+nr,k} = span_{\mathbb{C}}\{w(\mathbf{m},\mathbf{e}) \in V_M^{s+nr} \mid |w(\mathbf{m},\mathbf{e})| = k\}. \tag{4.27}$$

The statements (4.15) and (4.22) in Lemmas 2 and 4 imply that the action $U_0^{(s+nr)}$ and the commuting Hamiltonians preserve the degree:

$$U_0^{(s+nr)},\ h_l^{(s+nr)} : V_M^{s+nr,k} \to V_M^{s+nr,k}, \quad l \in \mathbb{Z}_{\neq 0},\ k = 0,1,\dots \quad . \tag{4.28}$$

Notice, that when $M = 0 \bmod n$ (i.e $s = 0$) the space $V_M^{s+nr,0}$ is one-dimensional with the basis $w(\mathbf{m}^0,\mathbf{e}^0)$.

Our main technical tool in defining the level-0 action in the space of semi-infinite wedges is the projection map $\rho^M_{r+1,r}$:

$$\rho^M_{r+1,r} : V_M^{s+nr+n} \to V_M^{s+nr}, \quad r = 0,1,\dots \quad , \tag{4.29}$$

which we define by specifying its action on the normally ordered wedges as follows:

Let $w(\mathbf{m},\mathbf{e}) \in V_M^{s+nr+n}$ and let

$$\mathbf{m} = (m_1, m_2, \dots, m_{s+nr}, m_{s+nr+1}, \dots, m_{s+nr+n}), \tag{4.30}$$

$$\mathbf{e} = (\epsilon_1, \epsilon_2, \dots, \epsilon_{s+nr}, \epsilon_{s+nr+1}, \dots, \epsilon_{s+nr+n}) \tag{4.31}$$

be the $\mathbf{m}$ and $\mathbf{e}$ sequences labeling the wedge $w(\mathbf{m},\mathbf{e})$.

Remove from $\mathbf{m}$ and $\mathbf{e}$ the last n elements, and denote the obtained sequences by $\mathbf{m}'$ and $\mathbf{e}'$:

$$\mathbf{m}' = (m_1, m_2, \ldots, m_{s+nr}), \tag{4.32}$$
$$\mathbf{e}' = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{s+nr}), \tag{4.33}$$

so that

$$w(\mathbf{m}', \mathbf{e}') \in V_M^{s+nr}.$$

The action of $\rho^M_{r+1,r}$ is then defined by:

$$\rho^M_{r+1,r}.w(\mathbf{m},\mathbf{e}) \quad = \quad \begin{cases} w(\mathbf{m}',\mathbf{e}') & \text{if } m_{s+nr+1} = m_{s+nr+2} = \cdots = m_{s+nr+n} = \\ & m^0_{s+nr+1} = m^0_{s+nr+2} = \cdots = m^0_{s+nr+n}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.34}$$

Proposition 5 *The following holds:*

$$\textit{The map } \rho^M_{r+1,r} \textit{ preserves the degree :} \qquad \rho^M_{r+1,r} : V_M^{s+nr+n,k} \to V_M^{s+nr,k}, \tag{i}$$

and for all $k = 0, 1, \ldots$ the map $\rho^{M,k}_{r+1,r} := \rho^M_{r+1,r}|_{V_M^{s+nr+n,k}}$ is surjective.

$$\textit{For } k \leq r \textit{ the map } \rho^{M,k}_{r+1,r} \textit{ is bijective.} \tag{ii}$$

Proof. The part (i) follows immediately from the definition of the degree (4.25) and from (4.34).

To prove the part (ii) let us demonstrate, that if $w(\mathbf{m},\mathbf{e}) \in V_M^{s+nr+n}$ is such that $|w(\mathbf{m},\mathbf{e})| \leq r$, then:

$$m_{s+nr+1} = m_{s+nr+2} = \cdots = m_{s+nr+n} = \\ = m^0_{s+nr+1} = m^0_{s+nr+2} = \cdots = m^0_{s+nr+n}. \tag{4.35}$$

Suppose the last equality does not hold. Then we necessarily have:

$$m_{s+nr+1} = m^0_{s+nr+1} - t_0, \quad \text{where } t_0 \geq 1. \tag{4.36}$$

Since the sequence $\mathbf{m}$ is n-strict and non-decreasing, we also have:

$$\begin{aligned} & m_{s+nr-n+1} < m_{s+nr+1}, \quad m^0_{s+nr-n+1} = m^0_{s+nr+1} - 1 \Rightarrow \\ & m_{s+nr-n+1} = m^0_{s+nr-n+1} - t_1, \quad t_1 \geq t_0, \end{aligned}$$

and in general:

$$m_{s+nr-nl+1} = m^0_{s+nr-nl+1} - t_l, \quad t_l \geq t_{l-1}, \quad l = 1, 2, \ldots, r.$$

Summing up the last equations for $l = 1, 2, \ldots, r$ and (4.36) we find:

$$|w(\mathbf{m},\mathbf{e})| = \sum_{i=1}^{s+nr+n} m^0_i - m_i \geq \sum_{l=0}^{r} t_l \geq t_0(r+1) \geq r+1.$$

This contradicts $|w(\mathbf{m},\mathbf{e})| \leq r$, and therefore (4.35) holds. Taking (4.34) into account we find that

$$Ker(\rho^M_{r+1,r}|_{V_M^{s+nr+n,k}}) = 0 \tag{4.37}$$

when $k \le r$. ■

An important property of the map $\rho^M_{r+1,r}$ (4.34) is that this map intertwines the $U'_q(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(s+nr+n)}$ defined in $V_M^{s+nr+n} \subset \wedge^{s+nr+n} V(z)$ with the $U'_q(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(s+nr)}$ defined in $V_M^{s+nr} \subset \wedge^{s+nr} V(z)$. The map $\rho^M_{r+1,r}$ also intertwines the actions of the operators $h_l^{(N)} - h_l^{(N)}(\mathbf{m}^0)I$ for $N = s+nr+n$ and $N = s+nr$ – in this case one needs to redefine $h_l^{(N)}$ (2.29) by subtracting the eigenvalue associated with the vacuum sequence (3.11) . We summarize this as the Proposition:

Proposition 6 *For $r = 0, 1, 2, \dots$ the following intertwining relations hold :*

$$\rho^M_{r+1,r} U_0^{(s+nr+n)} = U_0^{(s+nr)} \rho^M_{r+1,r}, \tag{i}$$

$$\rho^M_{r+1,r} g_l^{(s+nr+n)} = g_l^{(s+nr)} \rho^M_{r+1,r}, \quad l \in \mathbb{Z}_{\neq 0} \tag{ii}$$

where $g_l^{(N)} = h_l^{(N)} - h_l^{(N)}(\mathbf{m}^0)I$,

and $h_l^{(N)}(\mathbf{m}^0) = \sum_{j=1}^{N} p^{l m_j^0} q^{2l(1-j)}$.

To prove this Proposition we will need one more Lemma on properties of the operators (2.10):

Lemma 4 *Let $\mathbf{m} = (m_1, m_2, \dots, m_N) \in \mathbb{Z}^N$ be a sequence such, that:*

$$m_1, m_2, \dots, m_{N-k} < m_{N-k+1} = m_{N-k+2} = \dots = m_N \equiv m; \quad 1 \le k \le N. \tag{4.38}$$

Then for $a = \pm 1, \pm 2, \dots,$ the following relations hold:

$$\text{for } 0 \le l \le k-1 \qquad (Y_{N-l}^{(N)})^a \mathbf{z^m} = p^{am} q^{a(2k-2l-N-1)} \mathbf{z^m} + [\dots]\,, \tag{4.39}$$

$$\text{for } 1 \le i \le N-k \qquad (Y_i^{(N)})^a \mathbf{z^m} = q^{ak} (Y_i^{(N-k)})^a \mathbf{z^m} + [\dots]\,, \tag{4.40}$$

where $[\dots]$ signifies a linear combination of monomials $\mathbf{z^n} \equiv z_1^{n_1} z_2^{n_2} \dots z_N^{n_N}$ such that:

$$n_1, n_2, \dots, n_N \le m, \tag{4.41}$$

and

$$\#\{n_i | n_i = m\} < k. \tag{4.42}$$

Proof. Consider first the expression:

$$Y_{N-l}^{(N)} \mathbf{z^m} = \xi^{-1}_{N-l,N-l+1} \cdots \xi^{-1}_{N-l,N} p^{D_{N-l}} \xi_{1,N-l} \cdots \xi_{N-k,N-l} \cdot \\ \cdot \xi_{N-k+1,N-l} \cdots \xi_{N-l-1,N-l} \mathbf{z^m}, \quad 0 \le l \le k-1. \tag{4.43}$$

The eq. (4.17) gives:

$$Y_{N-l}^{(N)} \mathbf{z^m} = q^{k-l-1} \xi^{-1}_{N-l,N-l+1} \cdots \xi^{-1}_{N-l,N} p^{D_{N-l}} \xi_{1,N-l} \cdots \xi_{N-k,N-l} \mathbf{z^m}, \quad 0 \le l \le k-1. \tag{4.44}$$

In the last expression apply $\xi_{N-k,N-l}$ to the monomial $\mathbf{z^m}$ using the formula (4.16):

$$Y_{N-l}^{(N)} \mathbf{z^m} = q^{k-l-1} \xi^{-1}_{N-l,N-l+1} \cdots \xi^{-1}_{N-l,N} p^{D_{N-l}} \xi_{1,N-l} \cdots \xi_{N-k-1,N-l} (q^{-1} \mathbf{z^m} + [\dots]), \\ 0 \le l \le k-1, \tag{4.45}$$

where the meaning of $[\dots]$ is the same as in the statement of the Lemma.

Now apply $\xi_{i,N-l}$ repeatedly for $i = N-k-1, N-k-2, \dots, 1$ using at each step (4.13, 4.14) in Lemma 2 to show that $\xi_{i,N-l}([\dots]) = ([\dots])$, and using also (4.16). This gives:

$$Y^{(N)}_{N-l}\mathbf{z}^{\mathbf{m}} = q^{k-l-1}\xi^{-1}_{N-l,N-l+1}\dots\xi^{-1}_{N-l,N}(p^m q^{k-N}\mathbf{z}^{\mathbf{m}} + [\dots]), \quad 0 \le l \le k-1. \tag{4.46}$$

Lemma 2 and (4.17) applied in the last formula yield (4.39) for $a = 1$.

Consider now the expression:

$$Y^{(N)}_i\mathbf{z}^{\mathbf{m}} = \xi^{-1}_{i,i+1}\dots\xi^{-1}_{i,N-k}\xi^{-1}_{i,N-k+1}\dots\xi^{-1}_{i,N}p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}\mathbf{z}^{\mathbf{m}}, \quad 1 \le i \le N-k. \tag{4.47}$$

Write:

$$p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}\mathbf{z}^{\mathbf{m}} = (p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}z_1^{m_1}z_2^{m_2}\dots z_{N-k}^{m_{N-k}})z^m_{N-k+1}\dots z^m_N, \tag{4.48}$$

and observe that due to Lemma 2 the expression:

$$p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}z_1^{m_1}z_2^{m_2}\dots z_{N-k}^{m_{N-k}} \tag{4.49}$$

is a linear combination of monomials $z_1^{n_1}z_2^{n_2}\dots z_{N-k}^{n_{N-k}}$ such that $n_1, n_2, \dots, n_{N-k} < m$, and therefore the formula (4.16) implies that:

$$\xi^{-1}_{i,N}p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}\mathbf{z}^{\mathbf{m}} = q\,p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}\mathbf{z}^{\mathbf{m}} + [\dots]\,. \tag{4.50}$$

Continuing to apply Lemma 2 together with (4.16) we get :

$$\xi^{-1}_{i,N-k+1}\dots\xi^{-1}_{i,N}p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}\mathbf{z}^{\mathbf{m}} = q^k\,p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}\mathbf{z}^{\mathbf{m}} + [\dots]\,. \tag{4.51}$$

Finally we act by $\xi^{-1}_{i,i+1}\dots\xi^{-1}_{i,N-k}$ on the last expression and using Lemma 2 to show that $\xi^{-1}_{i,i+1}\dots\xi^{-1}_{i,N-k}([\dots])$ $= ([\dots])$, arrive at:

$$Y^{(N)}_i\mathbf{z}^{\mathbf{m}} = q^k\,\xi^{-1}_{i,i+1}\dots\xi^{-1}_{i,N-k}p^{D_i}\xi_{1,i}\dots\xi_{i-1,i}\mathbf{z}^{\mathbf{m}} + [\dots]\,, \quad 1 \le i \le N-k \tag{4.52}$$

which is the statement (4.40) of the Lemma at $a = 1$.

For $a = -1$ the proof is completely analogous, and (4.39,4.40) for $a = \pm 2, \pm 3, \dots$ follow from the case $a = \pm 1$ and Lemma 2, eqs. (4.13, 4.14). ∎

Proof of the Proposition 6 To prove the part (i) we consider the action of the generators of $U_0^{(s+nr+n)}$ on a wedge $w(\mathbf{m},\mathbf{e}) \in V_M^{(s+nr+n)}$.

First let $w(\mathbf{m},\mathbf{e})$ be such that:

$$m_i < m_i^0 \text{ for at least one } s+nr < i \le s+nr+n. \tag{4.53}$$

This condition is equivalent to $w(\mathbf{m},\mathbf{e}) \in Ker\rho^M_{r+1,r}$. The Lemmas 2 and 4 imply that acting with any of the generators (2.25 – 2.27) (where $y_i = q^{1-(s+nr+n)}Y_i^{(s+nr+n)}$) on such $w(\mathbf{m},\mathbf{e})$ produces a linear combination of wedges that have the property (4.53) as well, and therefore vanish when acted on by $\rho^M_{r+1,r}$. We formulate this as:

$$U_0^{(s+nr+n)} : Ker\rho^M_{r+1,r} \to Ker\rho^M_{r+1,r}. \tag{4.54}$$

Thus for $w(\mathbf{m},\mathbf{e})$ that satisfy (4.53) and any generator x of $U_q'(\widehat{\mathfrak{sl}}_n)$ one has:

$$x^{(s+nr)}\rho^M_{r+1,r}w(\mathbf{m},\mathbf{e}) = \rho^M_{r+1,r}x^{(s+nr+n)}w(\mathbf{m},\mathbf{e}) = 0. \tag{4.55}$$

Now let $w(\mathbf{m},\mathbf{e})$ be such that:

$$m_i = m_i^0 \text{ for all } s+nr < i \le s+nr+n\ . \tag{4.56}$$

With the same $\mathbf{m}'$ and $\mathbf{e}'$ as in (4.32, 4.33) one can write:

$$w(\mathbf{m},\mathbf{e}) = w(\mathbf{m}',\mathbf{e}') \wedge (u_{M-s-nr} \wedge u_{M-s-nr-1} \wedge \cdots \wedge u_{M-s-nr-n+1}), \tag{4.57}$$
$$\rho^M_{r+1,r} w(\mathbf{m},\mathbf{e}) = w(\mathbf{m}',\mathbf{e}'). \tag{4.58}$$

Apply $F_0^{(s+nr+n)}$ to $w(\mathbf{m},\mathbf{e})$:

$$F_0^{(s+nr+n)} w(\mathbf{m},\mathbf{e}) = \Lambda(\sum_{j=1}^{s+nr+n} q^{1-(s+nr+n)} Y_j^{(s+nr+n)} \mathbf{z}^{\mathbf{m}} \otimes (K_1^0)^{-1} \dots (K_{j-1}^0)^{-1} E_j^{1,n} \mathbf{v}_{\mathbf{e}}). \tag{4.59}$$

Lemma 4 for $a = 1$,$N = s+nr+n, k = n, m = m^0_{s+nr+n}$ and Lemmas 2, 4 enable us to transform the right-hand side of the last equation and arrive at:

$$\begin{aligned} F_0^{(s+nr+n)} w(\mathbf{m},\mathbf{e}) = (F_0^{(s+nr)} w(\mathbf{m}',\mathbf{e}')) \wedge (u_{M-s-nr} \wedge u_{M-s-nr-1} \wedge \cdots \wedge u_{M-s-nr-n+1}) + \\ + p^{m^0_{s+nr+r}} q^{2(1-nr-n-s)} ((K_0^{(s+nr)})^{-1} w(\mathbf{m}',\mathbf{e}')) \wedge \cdot \\ \cdot \wedge (u_{M-s-nr-n+1} \wedge u_{M-s-nr-1} \wedge u_{M-s-nr-2} \wedge \cdots \wedge u_{M-s-nr-n+1}) + \tilde{w}, \end{aligned} \tag{4.60}$$

where $\tilde{w} \in Ker \rho^M_{r+1,r}$. The normal ordering rules (2.23,2.24) imply that the second summand in the right-hand side of the last equation vanishes (cf. Lemma 2.2 in [1]). Finally (4.60) and (4.57,4.58) give:

$$F_0^{(s+nr)} \rho^M_{r+1,r} w(\mathbf{m},\mathbf{e}) = \rho^M_{r+1,r} F_0^{(s+nr+n)} w(\mathbf{m},\mathbf{e}). \tag{4.61}$$

Thus (i) is proven for the generator F_0. The rest of the $U'_q(\widehat{\mathfrak{sl}}_n)$-generators and the statement (ii) of the Proposition are handled in the same way. ∎

At fixed $M = s \bmod n$ and fixed degree k form the projective limit of the spaces $V_M^{s+nr,k}$ with respect to the map $\rho^{M,k}_{r+1,r} := \rho^M_{r+1,r}|_{V_M^{s+nr+n,k}}$:

$$V_M^k = \lim_{\substack{\longleftarrow \\ r}} V_M^{s+nr,k}. \tag{4.62}$$

A vector in V_M^k is a semi-infinite sequence $\{f_r\}_{r\geq 0}$; $f_r \in V_M^{s+nr,k}$ such that:

$$\rho^{M,k}_{r+1,r} f_{r+1} = f_r, \quad r = 0,1,2,\dots . \tag{4.63}$$

Since the map $\rho^{M,k}_{r+1,r}$ is bijective when $r \geq k$ we have the isomorphism of linear spaces:

$$V_M^k \cong V_M^{s+nr,k}, \quad r \geq k. \tag{4.64}$$

Notice that for $f = \{f_r\}_{r\geq 0} \in V_M^{s+nr,k}$, $r \geq k$:

$$f_{r+1} = f_r \wedge u_{M-s-nr} \wedge u_{M-s-nr-1} \wedge \cdots \wedge u_{M-s-nr-n+1}, \tag{4.65}$$

as implied by Proposition 5 (ii) and the definition (4.34) of the map $\rho^M_{r+1,r}$.

Now we use Propositions 5 (i) and 6 to define in the space V_M^k an $U'_q(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(\infty)}$, and an action of the commutaive family $\{g_l^{(\infty)}\}_{l\in\mathbb{Z}_{\neq 0}}$. For $f = \{f_r\}_{r\geq 0} \in V_M^k$ set:

$$U_0^{(\infty)}.\{f_r\} \;:=\; \{U_0^{(s+nr)}.f_r\}, \tag{4.66}$$
$$g_l^{(\infty)}.\{f_r\} \;:=\; \{g_l^{(s+nr)}.f_r\}. \tag{4.67}$$

Clearly we still have:

$$U_0^{(\infty)} g_l^{(\infty)} = g_l^{(\infty)} U_0^{(\infty)}, \quad l \in \mathbb{Z}_{\neq 0}. \tag{4.68}$$

The degree of a semi-infinite normally ordered wedge $w \in F_M$ is defined similarly to the degree (4.25) for wedges in V_M^{s+nr}. Write the vacuum vector $|M\rangle$ of F_M, and the w as:

$$|M\rangle = u_M \wedge u_{M-1} \wedge u_{M-2} \dots \quad = z^{m_1^0} v_{\epsilon_1^0} \wedge z^{m_2^0} v_{\epsilon_2^0} \wedge \dots \quad . \tag{4.69}$$

$$w = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \dots \quad ; \quad m_i = m_i^0, \epsilon_i = \epsilon_i^0 \text{ for } i >> 1 \ , \tag{4.70}$$

and define the degree $|w|$ as:

$$|w| = \sum_{i \geq 1} m_i^0 - m_i. \tag{4.71}$$

Let for $k \geq 0$:

$$F_M^k = \bigoplus_{\{\text{n.o. } w \in F_M \mid \, |w| = k\}} \mathbb{C} w \tag{4.72}$$

where "n.o." stands for "normally ordered". The Fock space is graded with respect to this degree: $F_M = \oplus_{k \geq 0} F_M^k$.

Define the map $\rho_M^k : V_M^k \to F_M^k$ by:

$$\text{for } f = \{f_r\}_{r \geq 0} \in V_M^k: \tag{4.73}$$

$$\rho_M^k f = f_r \wedge |M - s - nr\rangle, \quad \text{where } r \geq k. \tag{4.74}$$

Proposition 5 (or, equivalently the eq. (4.65)) shows that ρ_M^k does not depend on the choice of r in (4.74) as long as $r \geq k$.

The following Proposition will enable us to define a level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ in the Fock space.

Proposition 7 *The map ρ_M^k is an isomorphism of the linear spaces V_M^k and F_M^k for any $k \geq 0$.*

Proof. Since $V_M^{s+nr,k} \cong V_M^k$ for all $r \geq k$, it is sufficient to prove that the map:

$$V_M^{s+nk,k} \ni w^{(s+nk)} \longrightarrow w^{(s+nk)} \wedge |M - s - nk\rangle \in F_M \tag{4.75}$$

is an isomorphism of $V_M^{s+nk,k}$ and F_M^k.

Take a normally ordered wedge $w(\mathbf{m}, \mathbf{e}) \in V_M^{s+nk,k}$. This is an element of a basis of $V_M^{s+nk,k}$. Observe that the vector $w(\mathbf{m}, \mathbf{e}) \wedge |M - s - nk\rangle$ belongs to F_M^k and is a normally ordered wedge in F_M – that is an element of a basis in F_M. This shows injectivity of the map (4.75).

Let $w := z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \cdots$ be a normally ordered wedge in F_M^k. Applying the same reasoning as in the proof of the Proposition 5 (part ii) we can show, that

$$w \equiv w^{(s+nk)} \wedge |M - s - nk\rangle \tag{4.76}$$

where $w^{(s+nk)} \in V_M^{s+nk,k}$, and explicitely:

$$w^{(s+nk)} = z^{m_1} v_{\epsilon_1} \wedge z^{m_2} v_{\epsilon_2} \wedge \cdots \wedge z^{m_{s+nk}} v_{\epsilon_{s+nk}}. \tag{4.77}$$

Hence the map (4.75) is surjective. ■

Taking advantage of this Proposition we use the map ρ_M^k to define in F_M^k an $U_q'(\widehat{\mathfrak{sl}}_n)$-action U_0 along with an action of the commutative family $\{g_l\}$ by conjugating the actions $U_0^{(\infty)}$ and $g_l^{(\infty)}$ with the isomorphism ρ_M^k. Obviously we have $U_0 g_l = g_l U_0$ for all $l \neq 0$. The actions in $F_M = \oplus_{k \geq 0} F_M^k$ follow from the actions in each component F_M^k. This completes the definition of the level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ in the Fock space F_M.

Let us summarize the results of this section. We started from $U_0^{(N)}$ – the level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ in the finite wedge product $\wedge^N V(z)$. The generators $\{x^{(N)}\}$ ($x = E^a, F^a, K^a$, $a = 0, \ldots, n-1$) of this action are given by (2.25 - 2.27) with $y_i = q^{1-N} Y^{(N)}$. Using $U_0^{(N)}$ we have defined U_0 – level-0 action of $U_q'(\widehat{\mathfrak{sl}}_n)$ in the Fock space F_M . The space F_M is graded: $F_M = \oplus_{k\geq 0} F_M^k$ and by the definition the U_0 preserves the degree k. Any vector $w \in F_M^k$ ($M = s \bmod n$, $0 \leq s \leq n-1$) is represented as:

$$w = f_r \wedge |M - s - nr\rangle, \tag{4.78}$$

where $f_r \in V_M^{s+nr,k} \subset \wedge^{s+nr} V(z)$, and $r \geq k$. For any fixed $r \geq k$ this representation is unique by Proposition 7. The action of a generator $x \in U_q'(\widehat{\mathfrak{sl}}_n)$ on w (4.78) is then defined as:

$$x.w = (x^{(s+nr)}.f_r) \wedge |M - s - nr\rangle, \tag{4.79}$$

and $x.w$ does not depend on the choice of r as long as $r \geq k$ again by Proposition 7.

Similarly starting from the commutative family of operators $\{h_l^{(N)}\}$, $(l = \pm 1, \pm 2, \ldots)$ defined in $\wedge^N V(z)$ we define in F_M a commutative family of operators $\{g_l\}$, $(l = \pm 1, \pm 2, \ldots)$ which also commute with the U_0. As in (4.79) we prescribe the action of g_l on the wedge w (4.78) by:

$$g_l.w = (g_l^{(s+nr)}.f_r) \wedge |M - s - nr\rangle, \quad l = \pm 1, \pm 2, \ldots \quad ; \tag{4.80}$$

with $\{g_l^{(s+nr)}\}$ given in Proposition 6 (ii). The independence this prescription on the choice of the r is again due to the Proposition 7.

For computational purposes the most convenient choice of the r in (4.79, 4.80) is to take it to be minimal – that is $r = k$. We adopt this choice in the next section.

5 Decomposition of the Fock space with respect to the level-0 action

In this section we give the decomposition of the level-1 Fock space module of $U_q'(\widehat{\mathfrak{sl}}_n)$ with respect to the level-0 action U_0 which was defined in sec. 4.

5.1 Decomposition of the space V_M^{s+nr}

The definition of the $U_q'(\widehat{\mathfrak{sl}}_n)$-action U_0 given in sec. 4 makes it clear, that the decomposition of the Fock space will be found once we construct the decomposition of the spaces V_M^{s+nr} and $V_M^{s+nr,k}$ with respect to the $U_q'(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(s+nr)}$. To do this we use results of sec. 3. Let $N := s + nr$ and $E^{\mathbf{m}}$ be the subspace of $\wedge^N V(z)$ defined in (3.3). Recall that by Proposition 3 the $E^{\mathbf{m}}$ is an irreducible representation of the $U_0^{(s+nr)}$. Let $\mathbf{m}^0 \in \mathcal{M}_{s+nr}^n$ be the vacuum sequence (4.5) associated with the integer M. Then the $U_0^{(s+nr)}$-decomposition of V_M^{s+nr} is given by the following Proposition

Proposition 8 *For the $U_q'(\widehat{\mathfrak{sl}}_n)$ modules V_M^{s+nr} and $E^{\mathbf{m}}$ we have*

$$V_M^{s+nr} = \bigoplus_{\{\mathbf{m} \in \mathcal{M}_{s+nr}^n \mid m_{s+nr} \leq m_{s+nr}^0\}} E^{\mathbf{m}} \tag{5.1}$$

where the $U_q'(\widehat{\mathfrak{sl}}_n)$ action in the both sides is given by the $U_0^{(s+nr)}$.

Proof. We demonstrate that the set

$$B_M^{s+nr} := \{\phi(\mathbf{m}, \mathbf{e}) \equiv \Lambda(\Phi^{\mathbf{m}}(\mathbf{z}) \otimes \mathbf{v_e}) \mid \mathbf{m} \in \mathcal{M}_{s+nr}^n,\ m_{s+nr} \leq m_{s+nr}^0; \mathbf{e} \in \mathcal{E}(\mathbf{m})\} \tag{5.2}$$

is a base of the V_M^{s+nr}. First we note that $\phi(\mathbf{m},\mathbf{e}) \in B_M^{s+nr}$ implies $\phi(\mathbf{m},\mathbf{e}) \in V_M^{s+nr}$. This follows since the triangularity of the polynomial $\Phi^{\mathbf{m}}(\mathbf{z})$ and Lemma 4 allow us to represent the $\phi(\mathbf{m},\mathbf{e})$ as

$$\phi(\mathbf{m},\mathbf{e}) = w(\mathbf{m},\mathbf{e}) \quad + \sum_{\{\mathbf{n}\in\mathcal{M}^n_{s+nr}|\mathbf{n}^+\prec\mathbf{m}^+\}} \sum_{\mathbf{e}'\in\mathcal{E}(\mathbf{n})} c(\mathbf{m},\mathbf{e};\mathbf{n},\mathbf{e}')w(\mathbf{n},\mathbf{e}'). \tag{5.3}$$

From the last equation it is also follows that B_M^{s+nr} is a spanning set of V_M^{s+nr}. Finally the elements of B_M^{s+nr} are linearly independent by Proposition 1. By definition of the $E^{\mathbf{m}}$ (3.10) the set $\{\phi(\mathbf{m},\mathbf{e}) \mid \mathbf{e}\in\mathcal{E}(\mathbf{m})\}$ is a base of $E^{\mathbf{m}}$. Hence the result of the Proposition. ■

Since $E^{\mathbf{m}}$ is homogeneous with the degree $|\mathbf{m}|$, we have

Corollary 1

$$V_M^{s+nr,k} = \bigoplus_{\substack{\{\mathbf{m}\in\mathcal{M}^n_{s+nr} \mid m_{s+nr}\le m^0_{s+nr}\}\\ |\mathbf{m}^0|-|\mathbf{m}|=k}} E^{\mathbf{m}} \tag{5.4}$$

5.2 Decomposition of the Fock space

Let $M = s \bmod n$, $0\le s\le n-1$. Write the vacuum vector $|M\rangle \in F_M$ as

$$|M\rangle := u_M\wedge u_{M-1}\wedge\cdots \qquad \equiv z^{m^0_1}v_{\epsilon^0_1}\wedge z^{m^0_2}v_{\epsilon^0_2}\wedge\cdots \qquad . \tag{5.5}$$

The semi-infinite vacuum $\mathbf{m}$-sequence associated with the $|M\rangle$ is

$$\mathbf{m}^0 = (m^0_1, m^0_2,\cdots \quad) = (\underbrace{m^0,\ldots,m^0}_{s},\underbrace{m^0+1,\ldots,m^0+1}_{n},\underbrace{m^0+2,\ldots,m^0+2}_{n},\cdots \quad), \quad m^0 := \frac{s-M}{n}. \tag{5.6}$$

Introduce a set $\mathcal{M}^n[M]$ whose elements are ordered semi-infinite sequences $\mathbf{m}$ as follows

$$\mathcal{M}^n[M] := \{\mathbf{m} = (m_1\le m_2\le,\cdots) \mid \ \mathbf{m} \text{ is } n\text{-strict} ;\, m_i = m^0_i \text{ for } i>>1\}. \tag{5.7}$$

And for $\mathbf{m}\in\mathcal{M}^n[M]$ define the degree $\|\mathbf{m}\|$ as

$$\|\mathbf{m}\| := \sum_{i\ge1} m^0_i - m_i\ . \tag{5.8}$$

Note that $\|\mathbf{m}\|$ is a non-negative integer, and for a normally ordered wedge

$$w = z^{m_1}v_{\epsilon_1}\wedge z^{m_2}v_{\epsilon_2}\wedge z^{m_3}v_{\epsilon_3}\wedge\cdots \qquad (w\in F_M) \tag{5.9}$$

the degree $|w|$ (4.71) is equal to the $\|\mathbf{m}\|$.

Now for $\mathbf{m}\in\mathcal{M}^n[M]$ denote by $\mathbf{m}^{(N)}$ ($\in\mathcal{M}^n_N$) the ordered sequence obtained from the $\mathbf{m}$ by removing all except the first N elements.

Let for $\mathbf{m}\in\mathcal{M}^n[M]$ a linear space $\mathcal{F}^{\mathbf{m}}_M \subset F_M$ be defined in the following way

$$\mathcal{F}^{\mathbf{m}}_M := E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}} \wedge |M-s-n\|\mathbf{m}\|\rangle. \tag{5.10}$$

In the last formula we have

$$E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}} \subset V_M^{s+n\|\mathbf{m}\|,\|\mathbf{m}\|} \subset V_M^{s+n\|\mathbf{m}\|} \subset \wedge^{s+n\|\mathbf{m}\|}V(z), \quad \text{and} \quad \mathcal{F}^{\mathbf{m}}_M \subset F_M^{\|\mathbf{m}\|}. \tag{5.11}$$

The definition of the action U_0 given in sec. 4 and the Proposition 7 immediately lead to

Proposition 9 *For $\mathbf{m}\in\mathcal{M}^n[M]$ the $\mathcal{F}^{\mathbf{m}}_M$ is an $U'_q(\widehat{\mathfrak{sl}}_n)$-module with respect to the action U_0; the $U'_q(\widehat{\mathfrak{sl}}_n)$-modules $\mathcal{F}^{\mathbf{m}}_M$ with the $U'_q(\widehat{\mathfrak{sl}}_n)$-action U_0 and $E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}}$ with the $U'_q(\widehat{\mathfrak{sl}}_n)$-action $U_0^{(s+n\|\mathbf{m}\|)}$ are isomorphic.*

Note that since by Proposition 3 the $E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}}$ is irreducible, so is $\mathcal{F}_M^{\mathbf{m}}$.

Now observing that Corollary 1 and the Proposition 7 imply:

$$\bigoplus_{\substack{\mathbf{m}\in\mathcal{M}^n[M]\\ \|\mathbf{m}\|=k}} \mathcal{F}_M^{\mathbf{m}} = V_M^{(s+nk),k} \wedge |M-s-nk\rangle = F_M^k, \tag{5.12}$$

we obtain for the Fock space $F_M = \oplus_{k\geq 0} F_M^k$:

$$F_M = \bigoplus_{\mathbf{m}\in\mathcal{M}^n[M]} \mathcal{F}_M^{\mathbf{m}}. \tag{5.13}$$

This is the sought for decomposition of the Fock space with respect to the level-0 action U_0.

Since the structure of the $E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}}$ as a $U_q'(\widehat{\mathfrak{sl}}_n)$-module is known from Proposition 3 we can describe a component $\mathcal{F}_M^{\mathbf{m}}$ of the decomposition (5.13) by using the isomorphism of $E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}}$ and $\mathcal{F}_M^{\mathbf{m}}$.

To do this, by analogy with (3.20) for an $\mathbf{m}\in\mathcal{M}^n[M]$ introduce numbers r_k $(k=0,1,2,\dots)$ by

$$\mathbf{m} = (m_1 = \cdots = m_{r_1} < m_{1+r_1} = \cdots = m_{r_2} < m_{1+r_2} = \cdots = m_{r_3} < \quad \cdots \quad), \quad \text{and } r_0 := 0. \tag{5.14}$$

As the $U_q'(\widehat{\mathfrak{sl}}_n)$-module the space $\mathcal{F}_M^{\mathbf{m}}$ then is isomorphic to the semi-infinite tensor product:

$$\mathcal{F}_M^{\mathbf{m}} \cong V[p^{-m_{r_1}}q^{2r_1-2}, r_1] \otimes V[p^{-m_{r_2}}q^{2r_2-2}, r_2-r_1] \otimes \cdots \otimes V[p^{-m_{r_k}}q^{2r_k-2}, r_k-r_{k-1}] \otimes \quad \cdots \tag{5.15}$$

where $V[a,j]$ $(1\leq j\leq n)$ is the fundamental $U_q'(\widehat{\mathfrak{sl}}_n)$-module defined in (3.22). Note that since $m_i = m_i^0$ for all but finite number of elements in $\mathbf{m}$; we have $r_k - r_{k-1} = n$ for $k >> 1$, and hence the tensor product (5.15) contains only a *finite number* of factors different from the trivial 1-dimensional representation $V[a,n]$.

The Drinfel'd Polynomials of $\mathcal{F}_M^{\mathbf{m}}$ are found from the Drinfel'd Polynomials of the representation $E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}}$ (see Proposition 3), they are:

$$P_i(u) = \prod_{\{1\leq k<\infty \mid r_k - r_{k-1} = i\}} (u - p^{m_{r_k}} q^{-r_k - r_{k-1}}) \qquad (i\in\{1,2,\dots,n-1\}). \tag{5.16}$$

Let us remark that since $r_k - r_{k-1} = n$ for all sufficiently large k, the number of factors in the product above is always finite for any $\mathbf{m}\in\mathcal{M}^n[M]$.

Finally we find that the space $\mathcal{F}_M^{\mathbf{m}}$ is an eigenspace of the commuting Hamiltonians g_l ($l=\pm1,\pm2,\dots$) defined by (4.80) and the eigenvalue of g_l is

$$\sum_{i=1}^{\infty} (p^{lm_i} - p^{lm_i^0}) q^{2l(1-i)}. \tag{5.17}$$

Notice that the sum in the last expression contains only a finite number of non-zero summands due to the asymptotic condition $m_i = m_i^0$ for $i >> 1$.

An explicit base in the space $\mathcal{F}_M^{\mathbf{m}}$ ($\mathbf{m}\in\mathcal{M}^n[M]$) is immediately obtained from the definition (5.10) and a base of $E^{\mathbf{m}^{(s+n\|\mathbf{m}\|)}}$ described in (3.10).

A Appendix

In this appendix we summarize the conventions concerning the algebra $U_q'(\widehat{\mathfrak{sl}}_n)$ and the Drinfel'd Polynomials adopted in this paper. In particular we discuss the proof of Lemma 1.

A.1 Two realizations of $U'_q(\widehat{\mathfrak{sl}}_n)$

Let us recall the two realizations of $U'_q(\widehat{\mathfrak{sl}}_n)$ and the definition of the Drinfel'd Polynomials.

Definition 1 *[6] The quantum Kac–Moody algebra $U_q(g(A))$ associated to a symmetric generalized Cartan matrix $A=(a_{ij})_{i,j\in I:=\{0,1,\dots n-1\}}$ is the unital associative algebra over $\mathbb{C}$ with generators E_i, F_i, $K_i^{\pm 1}$ $(i\in I)$ and the following defining relations:*

$$\begin{aligned} K_iK_i^{-1} = \quad 1 \quad &= K_i^{-1}K_i, && \text{(A.1)}\\ K_iK_j \quad &= \quad K_jK_i\ , && \text{(A.2)}\\ K_iE_jK_i^{-1} \quad &= \quad q^{a_{ij}}E_j\ , && \text{(A.3)}\\ K_iF_jK_i^{-1} \quad &= \quad q^{-a_{ij}}F_j\ , && \text{(A.4)}\\ [E_i,F_j] \quad &= \quad \delta_{ij}\frac{K_i-K_i^{-1}}{q-q^{-1}}\ , && \text{(A.5)} \end{aligned}$$

$$\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_q(E_i)^rE_j(E_i)^{1-a_{ij}-r}=0\ ,\ i\neq j. \tag{A.6}$$

$$\sum_{r=0}^{1-a_{ij}}(-1)^r\begin{bmatrix}1-a_{ij}\\ r\end{bmatrix}_q(E_i)^rE_j(E_i)^{1-a_{ij}-r}=0\ ,\ i\neq j. \tag{A.7}$$

$$\textit{where } [n]_q:=\frac{q^n-q^{-n}}{q-q^{-1}},\quad \begin{bmatrix}n\\ r\end{bmatrix}_q:=\frac{[n]_q[n-1]_q\dots[n-r+1]_q}{[r]_q[r-1]_q\dots[1]_q}. \tag{A.8}$$

The coproduct Δ is given by

$$\begin{aligned} \Delta(E_i) \quad &:= \quad E_i\otimes K_i+1\otimes E_i, && \text{(A.9)}\\ \Delta(F_i) \quad &:= \quad F_i\otimes 1+K_i^{-1}\otimes F_i, && \text{(A.10)}\\ \Delta(K_i) \quad &:= \quad K_i\otimes K_i. && \text{(A.11)} \end{aligned}$$

In particular the algebra $U_q(\widehat{\mathfrak{sl}}_n)$ is the algebra $U_q(g(A))$, where the generalized Cartan matrix $A=(a_{ij})_{i,j\in I}$ is

$$a_{ij}=\begin{cases} 2 & (i=j)\\ -1 & (|i-j|=1,(i,j)=(1,n),(n,1))\\ 0 & (\text{otherwise})\end{cases}\quad n\geq 3, \tag{A.12}$$

$$a_{ij}=\begin{cases} 2 & (i=j)\\ -2 & (i\neq j)\end{cases}\quad n=2. \tag{A.13}$$

We put $c':=K_0K_1\dots K_{n-1}$ in $U_q(\widehat{\mathfrak{sl}}_n)$, then c' is the central in $U_q(\widehat{\mathfrak{sl}}_n)$. We define $U'_q(\widehat{\mathfrak{sl}}_n)$ as the quotient of $U_q(\widehat{\mathfrak{sl}}_n)$ by the two sided ideal generated by $c'-1$.

Proposition 10 *[6] $U'_q(\widehat{\mathfrak{sl}}_n)$ is isomorphic as an algebra to the algebra $\mathcal{A}$ with generators $E_{i,r}$, $F_{i,r}$ $(i\in\{1,\dots,n-1\},r\in\mathbb{Z})$, $H_{i,r}$ $(i\in\{1,\dots,n-1\}$, $r\in\mathbb{Z}\backslash\{0\})$, and $K_i^{\pm1}$, $(i\in\{1,\dots,n-1\})$, and the following*

defining relations:

$$
\begin{aligned}
K_iK_i^{-1} = \quad 1 \quad &= K_i^{-1}K_i, && \text{(A.14)}\\
K_iH_{j,r} \quad &= \quad H_{j,r}K_i\ , && \text{(A.15)}\\
[H_{i,r},H_{j,s}] \quad &= \quad 0\ , && \text{(A.16)}\\
K_iE_{j,r}K_i^{-1} \quad &= \quad q^{a_{ij}}E_{j,r}\ , && \text{(A.17)}\\
K_iF_{j,r}K_i^{-1} \quad &= \quad q^{-a_{ij}}F_{j,r}\ , && \text{(A.18)}\\
[H_{i,r},E_{j,s}] \quad &= \quad \frac{1}{r}[ra_{ij}]_qE_{j,r+s}\ , && \text{(A.19)}\\
[H_{i,r},F_{j,s}] \quad &= \quad -\frac{1}{r}[ra_{ij}]_qF_{j,r+s}\ , && \text{(A.20)}\\
E_{i,r+1}E_{j,s}-q^{a_{ij}}E_{j,s}E_{i,r+1} \quad &= \quad q^{a_{ij}}E_{i,r}E_{j,s+1}-E_{j,s+1}E_{i,r}, && \text{(A.21)}\\
F_{i,r+1}F_{j,s}-q^{-a_{ij}}F_{j,s}F_{i,r+1} \quad &= \quad q^{-a_{ij}}F_{i,r}F_{j,s+1}-F_{j,s+1}F_{i,r}, && \text{(A.22)}\\
[E_{i,r},F_{j,s}] \quad &= \quad \delta_{ij}\frac{\Phi^+_{i,r+s}-\Phi^-_{i,r+s}}{q-q^{-1}}\ , && \text{(A.23)}
\end{aligned}
$$

$$\sum_{\pi\in S_p}\sum_{k=0}^{p}(-1)^k\begin{bmatrix}p\\k\end{bmatrix}_q E_{i,r_{\pi(1)}}\dots E_{i,r_{\pi(k)}}E_{j,s}E_{i,r_{\pi(k+1)}}\dots E_{i,r_{\pi(p)}}=0\ ,\ i\neq j, \tag{A.24}$$

$$\sum_{\pi\in S_p}\sum_{k=0}^{p}(-1)^k\begin{bmatrix}p\\k\end{bmatrix}_q F_{i,r_{\pi(1)}}\dots F_{i,r_{\pi(k)}}F_{j,s}F_{i,r_{\pi(k+1)}}\dots F_{i,r_{\pi(p)}}=0\ ,\ i\neq j, \tag{A.25}$$

for all sequences $(r_1,\dots,r_p)\in\mathbb{Z}^p$, where $p=1-a_{ij}$ and the elements $\Phi^\pm_{i,r}$ are determined by equating coefficients of powers of u in the formal power series

$$\Phi_i^\pm(u):=\sum_{r=0}^{\infty}\Phi^\pm_{i,\pm r}u^{\pm r}=K_i^{\pm1}exp(\pm(q-q^{-1})\sum_{s=1}^{\infty}H_{i,\pm s}u^{\pm s}). \tag{A.26}$$

The generators of $\mathcal{A}$ are called Drinfel'd generators.

The isomorphisms $\tilde{f}:U_q'(\widehat{\mathfrak{sl}}_n)\to\mathcal{A}$ are not determined uniquely. In this paper we fix one isomorphism $f:U_q'(\widehat{\mathfrak{sl}}_n)\to\mathcal{A}$

$$f(E_i)=E_{i,0},\quad f(F_i)=F_{i,0},\quad f(K_i^{\pm1})=K_i^{\pm1}, \tag{A.27}$$

for $i\in\{1,\dots,n-1\}$, and

$$f(K_0^{\pm1})=(K_1K_2\dots K_{n-1})^{\mp1}, \tag{A.28}$$

$$\begin{gathered}f(E_0)=(-1)^{m-1}q^{-(n-2)/2}[F_{n-1,0},[F_{n-2,0},\dots \\ \dots,[F_{m+1,0},[F_{1,0},\dots,[F_{m-1,0},F_{m,1}]_{q^{1/2}}\dots]_{q^{1/2}}f(K_0),\end{gathered} \tag{A.29}$$

$$\begin{gathered}f(F_0)=\mu f(K_0^{-1})[E_{n-1,0},[E_{n-2,0},\dots \\ \dots,[E_{m+1,0},[E_{1,0},\dots,[E_{m-1,0},E_{m,-1}]_{q^{1/2}}\dots]_{q^{1/2}},\end{gathered} \tag{A.30}$$

where $\mu\in\mathbb{C}^\times$ is determined by

$$[f(E_0),f(F_0)]=\frac{f(K_0)-f(K_0^{-1})}{q-q^{-1}}, \tag{A.31}$$

and $[a,b]_{q^{1/2}}:=q^{1/2}ab-q^{-1/2}ba$.

A.2 Drinfel'd Polynomials

Now we will calculate the Drinfel'd Polynomials.

Let W be the representation of $U'_q(\widehat{\mathfrak{sl}}_n)$. W is said to be of type 1 if

$$W = \bigoplus_{\mu \in \mathbb{Z}^n} W_\mu, \tag{A.32}$$

where $W_\mu = \{w \in W | k_i.w = q^{\mu(i)} w\}$.

Proposition 11 *[6] Let W be a finite-dimensional irreducible $U'_q(\widehat{\mathfrak{sl}}_n)$–module of type 1. Then,*

(a) W is generated by a vector w_0 satisfying

$$E_{i,r}.w_0 = 0, \quad \Phi^{\pm}_{i,r}.w_0 = \phi^{\pm}_{i,r} w_0 \tag{A.33}$$

for all $i \in \{1, \dots, n\}$, $r \in \mathbb{Z}$, and some $\phi^{\pm}_{i,r} \in \mathbb{C}$.

(b) There exist unique monic polynomials $P_1(u), \dots, P_{n-1}(u)$ (depending on W) such that the $\phi^{\pm}_{i,r}$ satisfy

$$\sum_{r=0}^{\infty} \phi^{+}_{i,r} u^r = q^{\deg P_i} \frac{P_i(q^{-2}u)}{P_i(u)} = \sum_{r=0}^{\infty} \phi^{-}_{i,r} u^{-r}, \tag{A.34}$$

in the sense that the left and right-hand sides are the Laurent expansions of the middle term about 0 and ∞ respectively. Assigning to W the corresponding $n-1$–tuple of polynomials defines a one to one correspondence between the isomorphism classes of finite–dimensional irreducible $U'_q(\widehat{\mathfrak{sl}}_n)$–modules of type 1 and the set of $n-1$–tuples of monic polynomials in one variable u. We define the polynomials $P_1(u), \dots, P_{n-1}(u)$ to be the Drinfel'd Polynomials.

Remark If we change the isomorphism of the Proposition 10, the Drinfel'd Polynomials may be changed.

As a consequence of this Proposition, we get

Corollary 2 *[6] Let W be a finite-dimensional irreducible representation of $U'_q(\widehat{\mathfrak{sl}}_n)$ with associated polynomials P_i. Set $\lambda = (\deg\, P_1, \dots, \deg\, P_n)$. Then W contains the irreducible $U_q(\mathfrak{sl}_n)$–module $V(\lambda)$ with multiplicity one. Further, if $V(\mu)$ is any other $U_q(\mathfrak{sl}_n)$–module occurring in W, then $\lambda \succeq \mu$.*

Let Λ_j be the j-th fundamental weight of $\mathfrak{sl}_n$, then $V(\Lambda_j)$ is an irreducible representation of $U_q(\mathfrak{sl}_n)$. If $V(\Lambda_j)$ is also the representation of $U'_q(\widehat{\mathfrak{sl}}_n)$ ($\supset U_q(\mathfrak{sl}_n)$), then $V(\Lambda_j)$ is irreducible as a $U'_q(\widehat{\mathfrak{sl}}_n)$-module and by the Corollary 2, and the Drinfel'd Polynomials of $V(\Lambda_j)$ is

$$P_i(u) = \begin{cases} u - \tilde{a} & \text{if } i = j, \\ 1 & \text{otherwise,} \end{cases} \tag{A.35}$$

for some constant $\tilde{a}$. We define the representation determined by (A.35) as $V(\Lambda_j; \tilde{a})$.

We need the following Lemma.

Lemma 5 *[6] Let v_{Λ_j} be the $U_q(\mathfrak{sl}_n)$–highest weight vector in $V(\Lambda_j; \tilde{a})$, where $m \in \{1, \dots, n-1\}$, $\tilde{a} \in \mathbb{C}^\times$. Then,*

$$E_0.v_{\Lambda_j} = (-1)^{j-1} q^{-1} \tilde{a}^{-1} F_{n-1} F_{n-2} \dots F_{j+1} F_1 \dots F_j.v_{\Lambda_j}. \tag{A.36}$$

Remark This Lemma and the Lemma 6.4 in the paper [6] are different because the isomorphism between the realization of the Chevalley generators and the realization of the Drinfel'd generators are different.

proof Using the fact that the weight spaces of $V(\Lambda_j, \tilde{a})$ as a $U_q(\mathfrak{sl}_n)$–module are all one–dimensional and $K_i F_{j,1} K_i^{-1} = q^{-a_{ij}} F_{j,1}$, we get

$$F_{j,1}.v_{\Lambda_j} = b F_j.v_{\Lambda_j} \tag{A.37}$$

for some $b \in \mathbb{C}$. From the relation (A.23), we get

$$\Phi^+_{j,1}.v_{\Lambda_j} = b(q - q^{-1})v_{\Lambda_j}. \tag{A.38}$$

Hence, from the definition of the Drinfel'd Polynomials, we have

$$q(q^{-2}u - \tilde{a}) = (u - \tilde{a})(q + b(q - q^{-1})u + O(u^2)), \tag{A.39}$$

so that $b = \tilde{a}^{-1}$. Finally, from the relation (A.29), we find that

$$E_0.v_{\Lambda_j} = (-1)^{j-1}q^{-1}\tilde{a}^{-1}F_{n-1}F_{n-2}\dots F_{j-1}F_1\dots F_j.v_{\Lambda_j}. \tag{A.40}$$

■

From now on, we calculate the Drinfel'd Polynomials of $V[a,j]$ (3.22). $V[a,j]$ is the highest weight representation as a $U_q(\mathfrak{sl}_n)$-module, and the highest weight is Λ_j (the j-th fundamental weight). First we can check that $[v_1 \otimes v_2 \otimes \dots \otimes v_j]$ is the highest weight vector of $V[a,j]$ as a $U_q(\mathfrak{sl}_n)$–module. Because of the argument before the Lemma 5, the Drinfel'd Polynomials of $V[a,j]$ are

$$P_i(u) = \begin{cases} u - \tilde{a} & \text{if } i = j, \\ 1 & \text{otherwise,} \end{cases} \tag{A.41}$$

for some constant $\tilde{a}$. To determine $\tilde{a}$, we observe how the Chevalley generators act.

$$\begin{aligned} & E_0 \cdot ([v_1 \otimes v_2 \otimes \dots \otimes v_j]) \\ &= [E_0v_1 \otimes K_0v_2 \otimes \dots \otimes K_0v_j] + [v_1 \otimes E_0v_2 \otimes K_0v_3 \otimes \dots \otimes K_0v_j] + \dots \\ &= a[v_n \otimes v_2 \otimes \dots \otimes v_j]. \end{aligned} \tag{A.42}$$

$$\begin{aligned} & F_{n-1}\dots F_{j+1}F_1\dots F_j \cdot ([v_1 \otimes v_2 \otimes \dots \otimes v_j]) \\ &= F_{n-1}\dots F_{j+1}F_1\dots F_{j-1} \cdot (\dots + [K_j^{-1}v_1 \otimes \dots \otimes K_j^{-1}v_{j-1} \otimes F_jv_j]) \\ &= \dots = F_{n-1}\dots F_{j+1} \cdot ([v_2 \otimes v_3 \otimes \dots \otimes v_{j+1}]) \\ &= F_{n-1}\dots F_{j+2} \cdot (\dots + [K_{j+1}^{-1}v_2 \otimes K_{j+1}^{-1}v_3 \otimes \dots \otimes F_{j+1}v_{j+1}]) \\ &= \dots = ([v_2 \otimes \dots \otimes v_j \otimes v_n]). \end{aligned} \tag{A.43}$$

On the other hand we can check

$$v_k \otimes v_l + qv_l \otimes v_k \in Ker(q + S^{-1}) \ \text{ for } k < l. \tag{A.44}$$

Then

$$\begin{aligned} & [v_2 \otimes \dots \otimes v_j \otimes v_n] = (-q)[v_2 \otimes \dots v_{j-1} \otimes v_n \otimes v_j] \\ &= (-q)^2[v_2 \otimes \dots \otimes v_n \otimes v_{j-1} \otimes v_j] = \dots = (-q)^{j-1}[v_n \otimes v_2 \otimes \dots \otimes v_j]. \end{aligned} \tag{A.45}$$

From the Lemma 5, we get $\tilde{a} = q^{j-2}a^{-1}$. So we have proved the Lemma 1.

Next we verify the Proposition 3. Basically we use the following Proposition.

Proposition 12 *[6] If V, W, $V \otimes W$ are all irreducible as a $U'_q(\widehat{\mathfrak{sl}}_n)$–module, whose Drinfel'd Polynomials are $P_{V,i}(u)$, $P_{W,i}(u)$, $P_{V\otimes W,i}(u)$ $(i \in \{1, 2, \dots, n-1\})$, then*

$$P_{V\otimes W,i}(u) = P_{V,i}(u) \cdot P_{W,i}(u). \tag{A.46}$$

Using the fact written in the paper [8], we can check that our representation $W^{\mathbf{m}}$ (3.21) is irreducible if $\alpha \in \mathbb{R} \setminus \mathbb{Q}_{\leq 0}$ $(p = q^{-2\alpha})$. We explain the reason briefly. Let Λ_{k_i} be the k_i-th fundamental weight of $\mathfrak{sl}_n$. The key propositions are

∘ $V_1(\Lambda_{k_1}; a_1) \otimes V_2(\Lambda_{k_2}; a_2)$ is irreducible if and only if the intertwiner $R(V_1, V_2) : V_1(\Lambda_{k_1}; a_1) \otimes V_2(\Lambda_{k_2}; a_2) \to V_2(\Lambda_{k_2}; a_2) \otimes V_1(\Lambda_{k_1}; a_1)$ and $R(V_2, V_1)$ has no pole.

∘ $V_1(\Lambda_{k_1}; a_1) \otimes \dots \otimes V_l(\Lambda_{k_l}; a_l)$ is irreducible if and only if $V_i(\Lambda_{k_i}; a_i) \otimes V_j(\Lambda_{k_j}; a_j)$ is irreducible for all $i < j$.

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